

Title: When do substitution Delone sets have the Meyer property?

(Joint-work with Boris Solomyak)

Def  $\Lambda$  is a Meyer set if  $\Lambda$  is relatively dense and  $\Lambda - \Lambda$  is uniformly discrete.

Ex 1. (Meyer set)

- $\Lambda = (1 + 2\mathbb{Z}) \cup S$  for any subset  $S \subset 2\mathbb{Z}$ .

- $\Lambda = \{a + b\tau \in \mathbb{Z}[\tau] \mid a - b \cdot \frac{1}{\tau} \in [0, 1]\}$ , where  $\tau^2 - \tau - 1 = 0$ .

2. (Non-Meyer set)

$$\Lambda = \{n + \frac{1}{n} : n \in \mathbb{Z} \setminus \{0\}\}.$$

< Various known properties with Meyer sets >

Thm (Lagarias '95, Moody '97) characterizations of Meyer sets.

Thm (Baake - Moody '04)

If  $\Lambda$  is a Meyer set admitting autocorrelation,

then  $\Lambda$  is pure point diffractive if and only if

for any  $\varepsilon > 0$ ,  $\{t \in \mathbb{R}^d : \text{density}(\Lambda \cap (\Lambda - t)) < \varepsilon\}$  is relatively dense.

Thm (Strungaru '05)

If  $\Lambda$  is a Meyer set with uniform cluster frequencies, then

the Bragg peaks in the diffraction pattern of  $\Lambda$  are relatively dense.

It implies that the set of eigenvalues for the dynamical system  $(X_\Lambda, \mathbb{R}^d, \mu)$  is

relatively dense.

Thm (Akiyama - Lee '10)

If a substitution point set is a Meyer set, then one can determine

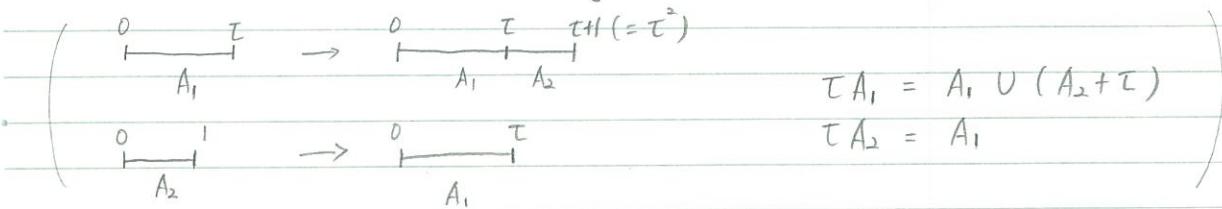
pure point spectrum using a computational algorithm.

- If there exist an expansive map  $\phi$ , non empty compact set  $A_i$ 's with  $A_i = \overline{A_i}$ , and finite sets  $D_{ij}$  such that

$$\phi A_j = \bigcup_{i \leq m} (A_i + D_{ij}),$$

where all sets in the right hand side have disjoint interiors, then a tiling constructed with  $A_i$ 's is called a substitution tiling.

Ex (Fibonacci substitution tiling)



Let  $\tau$  be a substitution tiling in  $\mathbb{R}^d$  with an expansion map  $\phi$  which has finite local complexity (FLC) and  $\Lambda_\tau = (\Lambda_i)_{i \in m}$  be a substitution point set representing  $\tau$ .

Let  $\Xi = \{x \in \mathbb{R} : \exists T, T-x \in \tau\}$  and  $K = \{x \in \mathbb{R}^d : \tau = \tau - x\}$ .

Question

How can we determine the Meyer property on substitution tilings?

Thm (Lee - Solomyak '08)

The set of eigenvalues for  $(X_\tau, \mathbb{R}^d, \mu)$  is relatively dense if and only if the corresponding substitution point set  $\Lambda_\tau$  is a Meyer set.

Thm (Solomyak '06)

$\tau$  is an eigenvalue for  $(X_\tau, \mathbb{R}^d, \mu)$  if and only if

$$\lim_{n \rightarrow \infty} e^{2\pi i \langle \phi^n x, \tau \rangle} = 1 \quad \text{for all } x \in \Xi,$$

$$e^{2\pi i \langle x, \tau \rangle} = 1 \quad \text{for all } x \in K = \{x \in \mathbb{R}^d : \tau - x = \tau\}.$$

Thm (Kenyon '94, Solomyak '06)

Let  $\phi$  be a similarity map in  $\mathbb{R}^d$  such that  $\phi = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ , where  $|\lambda| > 1$ .

Then  $\Xi \subset \mathbb{Z}[\lambda]d_1 + \dots + \mathbb{Z}[\lambda]d_d$  for some basis  $\{d_1, \dots, d_d\}$  of  $\mathbb{R}^d$ .

(Question what if  $\phi$  is not a similarity map? In other words, what if the eigenvalues of  $\phi$  are not all same?)

Thm (Lee - Solomyak '10)

Suppose that  $\phi$  is diagonalizable and all the eigenvalues of  $\phi$  are algebraically conjugate with the same multiplicity  $m$ .

Then  $\exists$  an isomorphism  $\rho: \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that

$$\rho\phi = \phi\rho \text{ and } \Xi \subset \rho(\mathbb{Z}[\phi]d_1 + \dots + \mathbb{Z}[\phi]d_J),$$

$$\text{where } (d_j)_n = \begin{cases} 1 & \text{if } (j-1)m+1 \leq n \leq jm \\ 0 & \text{else} \end{cases}$$

$$\text{and } J \cdot m = d. \quad \text{Fix if } \phi = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix}, \text{ then } d_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, d_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \dots$$

We show now how this theorem is used to get the Meyer property of  $\Xi$ .

- An algebraic integer  $\lambda$  is a Pisot number if  $|\lambda| > 1$  and all other

- algebraic conjugates are less than 1 in modulus.

- A set  $\Lambda = \{\lambda_1, \dots, \lambda_m\}$  of algebraic integers is a Pisot family if

- for every  $\lambda_i \in \Lambda$ , if  $r$  is an algebraic conjugate of  $\lambda_i$  and  $r \notin \Lambda$ , then

$$|r| < 1.$$

Lemma Let  $\lambda$  be a Pisot number.

Then  $\text{dist}(\lambda^n, \mathbb{Z}) \rightarrow 0$  as  $n \rightarrow \infty$ .

Pf> Let  $\lambda_2, \dots, \lambda_s$  be all the algebraic conjugates of  $\lambda$ .

For any  $n \in \mathbb{Z}_+$ ,  $\lambda^n + \sum_{j=2}^s (\lambda_j)^n \in \mathbb{Z}$ .

Note that  $|\sum_{j=2}^s (\lambda_j)^n| \leq (s-1) \sup_{2 \leq j \leq s} |\lambda_j|^n \rightarrow 0$  as  $n \rightarrow \infty$ .

Thus the claim follows.

Lemma Let  $\lambda = \{\lambda_1, \dots, \lambda_m\}$  be a Pisot family.

Then  $\text{dist}(\sum_{k=1}^m (\lambda_k)^n, \mathbb{Z}) \rightarrow 0$  as  $n \rightarrow \infty$ .

prop If the set of eigenvalues of  $\phi$  is a Pisot family, then

the set of eigenvalues for  $(X_T, \mathbb{R}^d, \mu)$  is relatively dense.

Pf> To be simple, we assume in this talk that all the eigenvalues of  $\phi$  are real.

For any  $n \in \mathbb{Z}_{\geq 0}$  and  $0 \leq l < m$ ,

$$\langle \phi^n d_j, (\phi^T)^l d_j \rangle = \langle \phi^{n+l} d_j, d_j \rangle = \sum_{k=1}^m \lambda_k^{n+l}.$$

Since  $\{\lambda_1, \dots, \lambda_m\}$  is a Pisot family,  $\text{dist}(\sum_{k=1}^m \lambda_k^{n+l}, \mathbb{Z}) \rightarrow 0$  as  $n \rightarrow \infty$ .

Note  $\langle \phi^n d_i, (\phi^T)^l d_j \rangle = 0$  if  $i \neq j$ .

Hence  $\lim_{n \rightarrow \infty} \bigcup_{i=1}^{2\pi i} \langle \phi^n y, (\phi^T)^l d_j \rangle = 1$  for all  $y \in \mathbb{Z}[d_1, \dots, d_m]$ .

Thus  $\lim_{n \rightarrow \infty} \bigcup_{i=1}^{2\pi i} \langle \phi^n x, (\phi^T)^l d_j \rangle = 1$  for all  $x \in \mathbb{Z}$ .

From uniform convergence of the limit in  $x \in \mathbb{Z}$ ,

$$\bigcup_{i=1}^{2\pi i} \langle x, (\phi^T)^l (\phi^T)^{K+l} d_j \rangle = 1 \text{ for all } x \in \mathbb{X} \text{ and some big } K \in \mathbb{Z}_+.$$

So  $(\phi^T)^l (\phi^T)^{K+l} d_j$  is an eigenvalue for  $(X_T, \mathbb{R}^d, \mu)$  for  $l = 0, \dots, m-1$ .

Since  $\{d_1, \dots, (\phi^T)^{m-1} d_1, \dots, d_J, \dots, (\phi^T)^{m-1} d_J\}$  is a basis of  $\mathbb{R}^d$ , the claim follows.

Thm (Vijayaraghavan '40)

Let  $u_1, u_2, \dots$  be a sequence of real numbers, where

$$u_n = c_1 \lambda_1^n + c_2 \lambda_2^n + \dots + c_m \lambda_m^n, \quad c_1, c_2, \dots, c_m \neq 0,$$

$\lambda_1, \dots, \lambda_m$  are distinct algebraic numbers, and  $|\lambda_k| > 1$  ( $k = 1, \dots, m$ ).

If  $\text{dist}(u_n, \mathbb{Z}) \rightarrow 0$  as  $n \rightarrow \infty$ , then  $\{\lambda_1, \dots, \lambda_m\}$  is a Pisot family.

Prop If  $\gamma$  is a non-zero eigenvalue for  $(X_\gamma, \mathbb{R}^d, \mu)$ , then

the set of eigenvalues of  $\phi$  is a Pisot family.

Pf For any  $x \in \mathbb{Z}$ ,  $x = p \left( \sum_{j=1}^J g_j(\phi) d_j \right)$  for some polynomials  $g_j \in \mathbb{Z}[x]$ .

$$\langle \phi^n x, \gamma \rangle = \sum_{j=1}^J \langle \phi^n g_j(\phi) d_j, \gamma^j \rangle$$

$$= \sum_{k=1}^m c_k \lambda_k^n \quad \text{for some } c_k \in \mathbb{C}$$

Since  $\gamma$  is an eigenvalue,  $\text{dist}(\langle \phi^n x, \gamma \rangle, \mathbb{Z}) \rightarrow 0$  as  $n \rightarrow \infty$ .

By Vijayaraghavan's thm, the set of eigenvalues of  $\phi$  is a Pisot family.

Thm TFAE

(1) The set of eigenvalues of  $\phi$  forms a Pisot family

(2) The set of eigenvalues of  $(X_\gamma, \mathbb{R}^d, \mu)$  is relatively dense

(3)  $(X_\gamma, \mathbb{R}^d, \mu)$  is not weakly mixing

(4)  $\mathbb{Z}$  is a Meyer set.  
( $\Lambda_\gamma$ )