On a beta expansion with negative bases

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<u>CONTENTS</u>

§1 Beta expansion

(1) Definition

$x \in (0,1)$, $\beta > 1$

$$x = \frac{a_1(x)}{\beta} + \frac{a_2(x)}{\beta^2} + \cdots$$

where

$$\begin{aligned} a_n(x) &:= [\beta T_\beta^{n-1}(x)] \in \{0, 1, \cdots, [\beta]\} \\ T_\beta(x) &:= \{\beta x\} : [0,1) \to [0,1) \end{aligned}$$

(2) Admissible sequence

Def.

A sequence $(x_n)_{n\geq 1}$ **is** β- *admissible* $\overset{def}{\Longleftrightarrow}$ $(x_n)_{n\geq 1}$ *is the* β*-expansion of some* $x \in [0,1)$.

Notation

$$d(x,\beta) := (a_1(x), a_2(x), \cdots)$$
$$d^*(1,\beta) := \lim_{\epsilon\downarrow 0} d(1-\epsilon,\beta)$$

Theorem(Parry)

$(x_n)_{n\geq 1}$ **is** β**-admissible**

$$\Longleftrightarrow \quad (x_m, x_{m+1}, \cdots) \preceq_{lex} d^*(1,\beta),\ \forall m \geq 1$$

Ex. $\beta = \tau$**,** $d^*(1,\tau) = (1,0,1,0,\cdots)$**.**

(3) Shift Space

Def.

$$X_\beta := \{(x_n)_{n\in\mathbf{Z}} \mid \ \forall \text{ subword of } (x_n) \text{ appears in an admissible seq. } \}$$

Theorem

(1)(Ito-Takahashi 1974)
X_β **is SFT**
$\iff d^*(1,\beta)$ **is purely-periodic**

(2)(Bertrand-Mathis 1986)
X_β **is Sofic**
$\iff d^*(1,\beta)$ **is eventually-periodic**

Ex.
$\beta = \tau$, $X_\tau = \{(x_n)_{n\in\mathbf{Z}} \mid 1 \text{ is isolated } \}$.

(4) Invariant measure

The invariant measure on $[0, 1)$ w.r.t. T_β is given by

$$\nu(E) = \int_E h_\beta(x)dx,$$

where

$$h_\beta(x) = \sum_{n \geq 0} 1_{\{x < T_\beta^n(1)\}} \cdot \beta^{-n}$$
$$T_\beta^n(1) := T_\beta^{n-1}((\beta)),\ T_\beta^0(1) := 1$$

Normalization Constant

$$F(\beta) := \int_0^1 h_\beta(x)dx$$

Theorem(Parry)

(1) $F(\beta)$ is right continuous
(2) $F(\beta)$ is left *dis*-continuous at simple Parry numbers.

§2 Ito-Sadahiro expansion
(Ito-Sadahiro, 2009)

(1) Definition

<u>Def.</u>

$$x \in (l_\beta, r_\beta), \quad l_\beta := \frac{-\beta}{\beta+1},\ r_\beta := \frac{1}{\beta+1},\ \beta > 1$$

$$x = \frac{a_1(x)}{(-\beta)} + \frac{a_2(x)}{(-\beta)^2} + \cdots$$

where

$$a_n(x) = [-\beta T_{-\beta}^{n-1}(x) - l_\beta] \in \{0, 1, \cdots [\beta]\}$$
$$T_{-\beta}(x) = -\beta x - [-\beta x - l_\beta]$$

(2) Admissible seq.

Def.

$(x_n)_{n\geq 1}$ **is** $(-\beta)$**-admissible**
$\overset{def}{\Longleftrightarrow}$ (x_n) **is the** $(-\beta)$**-expansion of some** $x \in [l_\beta, r_\beta)$**.**

Notation

$$d(x, -\beta) = (a_1(x), a_2(x), \cdots)$$
$$d^*(r_\beta, -\beta) := \lim_{\epsilon \downarrow 0} d(r_\beta - \epsilon, -\beta)$$

Theorem

$(x_n)_{n\geq 1}$ **is** $(-\beta)$**-admissible**

$$\Longleftrightarrow$$
$$d(l_\beta, -\beta) \preceq_{IS} (x_m, x_{m+1}, \cdots),$$
$$(x_m, x_{m+1}, \cdots) \preceq_{IS} d^*(r_\beta, -\beta)$$

for $\forall m \geq 1$**, where**

$$(a_n)_{n\geq 1} \preceq_{IS} (b_n)_{n\geq 1}$$
$$\overset{def}{\Longleftrightarrow} ((-1)^n a_n)_{n\geq 1} \preceq_{lex} ((-1)^n b_n)_{n\geq 1}$$

(3) Shift Space

Def.

$X_{-\beta} := \{(x_n)_{n\in\mathbf{Z}} \mid$ $\forall$ **subword of** (x_n) **appears in an** $(-\beta)$ **admissible seq.** $\}$

Theorem

$X_{-\beta}$ **is sofic**
$\Longleftrightarrow$ $d(l_\beta, -\beta)$ **is eventually periodic.**

Ex.

$\beta = \tau := (1+\sqrt{5})/2$
$d(l_\beta, -\tau) = (1, 0, 0, \cdots)$

The set of forbidden words is

$$\{1\underbrace{0\cdots0}_{\textbf{odd}}1\}$$

(4) Invariant Measure

Theorem

The invariant measure is

$$\nu(E) = \int_E h_{-\beta}(x)dx,$$

where

$$h_{-\beta}(x) = \sum_{n=0}^{\infty} 1_{\{T_{-\beta}^n(l_\beta) \le x\}} \cdot (-\beta)^{-n}.$$

Normalization Constant

$$F(-\beta) := \int_{l_\beta}^{r_\beta} h_{-\beta}(x)dx$$

$Q_{even} := \{\beta \mid d(l_\beta, -\beta)$ **is even periodic** $\}$
$Q_{odd} := \{\beta \mid d(l_\beta, -\beta)$ **is odd periodic** $\}$

Theorem

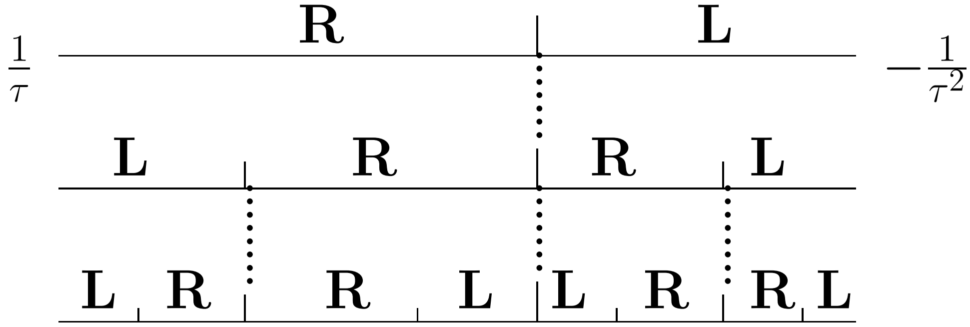
(1) At $\beta \in Q_{odd}$, $F(-\beta)$ is right-continuous, and left-discontinuous,
(2) At $\beta \in Q_{even}$, $F(-\beta)$ is left-continuous, and right-discontinuous.

§3 An $(-\beta)$-expansion related to Sturmian seq.

(1) Definition (for $\beta = \tau$)

(ext'd. to α's with $\alpha^2 + k\alpha = 1$, $\beta = \alpha^{-1}$, $k \in \mathbf{N}$).

$$x \in \left[-\frac{1}{\tau^2}, \frac{1}{\tau}\right]$$



$$\begin{aligned} x &\mapsto (R, L, R, L, \cdots) \quad (R \mapsto 1,\ L \mapsto 01) \\ &\mapsto (1, 0, 1, 1, 0, 1, \cdots) \\ &=: (a_2(x), a_3(x), \cdots) \end{aligned}$$

then $$x = \sum_{j \geq 2} \frac{a_j(x)}{(-\beta)^j}$$

Relation to Fibonacci words

$$v_\theta(n) = 1_{[1-\frac{1}{\tau},1)}\left(\frac{1}{\tau}\cdot n + \theta\right), \qquad \theta \in [0,1),\ n \in \mathbf{Z}.$$

R, L-construction of v_θ

$$s_{-1} = 1, \qquad s_0 = 0, \qquad s_1 = 1$$

$$s_{n+1} = s_n\ s_{n-1}$$

$$\boxed{s_n}$$
$$\downarrow R$$
$$\boxed{s_n}\boxed{s_{n-1}}$$
$$||$$
$$\boxed{\quad s_{n+1} \quad}$$

$$\boxed{s_n}$$
$$L \downarrow$$
$$\boxed{s_{n+1}}\boxed{s_n}$$
$$||$$
$$\boxed{\quad s_{n+2} \quad}$$

Any v_θ can be constructed in this manner.

$$v_\theta \mapsto (R, L, R, L, \cdots)$$

which has the same rep. for

$$x \mapsto (R, L, R, L, \cdots)$$

with

$$x + \frac{1}{\tau^2} = \theta$$

<u>Results</u>

(2) Admissible seq.

$(x_n)_{n\geq 1}$ **is** $(-\tau)$**-admissible**

$\Longleftrightarrow (0,1,0,1,\cdots) \preceq_{lex} (x_m, x_{m+1},\cdots), \forall m \geq 1,$

and (x_n) **does not have tail of** $011\overline{01}$**.**
$\Longleftrightarrow 0$ **is isolated and** (x_n) **does not have tail of** $011\overline{01}$**.**

(3) Shift Space

$(x_n)_{n\in\mathbf{Z}} \in X_{-\tau}$
$\Longleftrightarrow 0$ **is isolated**
$\Longleftrightarrow$ **The set of forbidden word is** $\{00\}$

Hence $X_{-\tau}$ **is SFT.**

(4) Invariant Measure

$$h_{-\tau}(x) = \begin{cases} \frac{1}{\tau} & (-\frac{1}{\tau^2} < x < 0) \\ 1 & (0 < x < \frac{1}{\tau}) \end{cases}$$

(5) Relation between two expansions

$$\frac{1}{(-\tau)^k} - \frac{1}{(-\tau)^{k+1}} = \frac{1}{(-\tau)^{k+2}}$$

Hence

$$\begin{array}{cccc} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ = & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{array} \qquad \cdots (*)$$

Any $(x_n)_{n=1}^{\infty}(\subset \{0,1\}^{\mathbf{N}})$ **can be "modified" via** $(*)$ **into both**

(i) IS -admissible seq.
and
(ii) Sturmian -admissible seq.

$$\begin{array}{rccc} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ = & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{array} \qquad \cdots (*)$$

<u>Ex.</u>

(1) IS $\rightarrow$ S

$$\begin{array}{ccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ & & & & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ & & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{array}$$

(2) S $\rightarrow$ IS

$$\begin{array}{cccccccccc} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{array}$$