

# Matrix integrals and free probability

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We consider integrals on unitary groups  $\mathbb{U}_d$  of the form

$$\int_{\mathbb{U}_d} U_{i_1 j_1} \cdots U_{i_q j_q} U_{j'_1 i'_1}^* \cdots U_{j'_q i'_q}^* dU$$

We give an explicit formula in terms of characters of symmetric groups and Schur functions, which allows us to rederive an asymptotic expansion as  $d \rightarrow \infty$ . Using this we rederive and strengthen a result of asymptotic freeness due to Voiculescu.

We then study large  $d$  asymptotics of matrix model integrals and of the logarithm of Itzykson-Zuber integrals and show that they converge towards a limit when considered as power series. In particular we give an explicit formula for

$$\lim_{d \rightarrow \infty} \frac{\partial^n}{\partial z^n} d^{-2} \log \int_{\mathbb{U}_d} e^{z \text{Tr}(XUYU^*)} dU|_{z=0}$$

assuming that the normalized traces  $d^{-1} \text{Tr}(X^k)$  and  $d^{-1} \text{Tr}(Y^k)$  converge in the large  $d$  limit. We consider as well a different scaling and relate its asymptotics to Voiculescu's R-transform.

## References

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