The Carathéodory convergence of Fatou components of polynomials to Baker domains or wandering domains II

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Abstract

We consider a sequence of polynomials which converges locally uniformly in $\mathbb{C}$ to a transcendental entire function whose Fatou set contains a Baker domain and wandering domains. We show that the sequence of the Fatou sets of the polynomials converges to the Fatou set of the transcendental entire functions in the Carathéodory sense.

1 Introduction

Let f be an entire functions. We denote the Fatou set of f and the Julia set of f by $F(f)$ and $J(f)$, respectively. In this note, if f is a transcendental entire function, then we regard $J(f)$ contains the point at infinity. There are several difference between the dynamics of polynomials and that of transcendental entire functions. For example, though wandering domains and Baker domains are never appeared in the iteration of polynomials, those may be appeared in case of transcendental entire functions. On the other hand, every transcendental entire functions can be approximated by sequences of polynomials in the sense of locally uniformly convergence in $\mathbb{C}$. The problem is to see whether the the sequence of the dynamics of polynomials converges to that of the transcendental entire functions or not. Devaney, Goldberg and Hubbard [3] studied the exponential family $z \mapsto \lambda e^z$ by approximating with $z \mapsto \lambda(1 + z/n)^n$ and showed the convergence of hyperbolic components and external rays in the parameter space. After their works, Krauskopf showed that if an exponential function has an attracting component, then the sequence of the Julia sets of approximating polynomials converges to the Julia

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set of the exponential function in the Hausdorff metric. Furthermore Kisaka proved if the Fatou set of a transcendental entire function consists only of basins of attracting cycles then the any sequence of the Julia sets of polynomials which converges to the function locally uniformly in $\mathbb{C}$ converges to the Julia set in the Hausdorff metric. In [9], some transcendental entire functions whose Fatou sets have wandering domains or Baker domains were treated. In this note, we consider a transcendental entire function which has a Baker domain and wandering domains. The way a sequence of Fatou components of polynomial converges to a wandering domain is different from that we showed in [9]. The function we considered here was given by Bergweiler [2].

Theorem 1 *Let $f(z) = 2 + 2z - 2e^z$ and*

$$f_n(z) = -2\left(1+\frac{z}{2^n}\right)^{2^n+2} + \left(1+\frac{1}{2^{n-1}}\right)\frac{1}{2^n}z^2 + 2\left(1+\frac{1}{2^{n-1}}\right)z + 2.$$

Then $\{f_n\}$ converges to f locally uniformly in $\mathbb{C}$ and $\{J(f_n)\}$ converges to $J(f)$ in the Hausdorff metric.

The reader are expected to be familiar with the basic results of complex dynamics of transcendental entire functions, which can be found in e.g. [8].

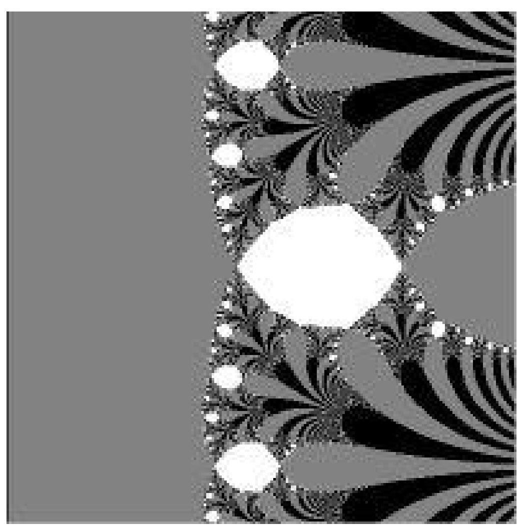

Figure 1: The Fatou set of the function $f(z) = 2 + 2z - 2e^z$. The Baker domain and its backward images are grey. The basin of the attraction of 0 and wandering domains are white. The Julia set is black.

2 Preliminaries

First, we introduce an idea of convergence of Julia sets. Let ρ be the chordal metric on $\widehat{\mathbb{C}}$. The ϵ-neighborhood of a set A is denoted by $U_\epsilon(A)$. Let A and B be non-empty compact sets in $\widehat{\mathbb{C}}$. The Hausdorff distance $d(A,\ B)$ of A and B is defined by

$$d(A,\ B) = \inf\{\varepsilon > 0 \mid A \subset U_\varepsilon(B), \quad B \subset U_\varepsilon(A)\}.$$

This distance defines the Hausdorff metric on the set of all the non-empty compact sets in $\widehat{\mathbb{C}}$. Let K and K_n $(n \in \mathbb{N})$ be non-empty compact sets in $\widehat{\mathbb{C}}$. We say that K_n converges to K in the Hausdorff metric if $d(K_n,\ K) \to 0$ as $n \to \infty$. Since the set of all cyclic points are dense in the Julia set, it is easy to see the following.

Proposition 2 *If f_n converges to f locally uniformly, then, for an arbitrary $\epsilon > 0$, there exists an N such that*

$$J(f) \subset U_\varepsilon(J(f_n))$$

for all $n \geq N$.

This proposition implies the Hausdorff convergence of Julia sets is *lower semicontinuous* (see Douady [4]).

Next, we introduce an idea of convergence of open sets. Let O and O_n $(n \in \mathbb{N})$ be open sets in $\widehat{\mathbb{C}}$. We say that O_n converges to O in the Carathéodory sense if the following two conditions hold:
(1) for an arbitrary compact set $I \subset O$, there exits an N such that $I \subset O_n$ for all $n \geq N$ and
(2) if an open set U is contained in O_m for infinitely many m, then $U \subset O$.

Two ideas of convergence defined above have the following relationship.

Lemma 3 *Non-empty closed set K_n converges to K in the Hausdorff metric if and only if the complement K_n^c of K_n converges to the complement K^c of K in the Carathéodory sense.*

Proposition 2 is rephrased as follows.

Lemma 4 *Assume f_n converges to f locally uniformly. If there exists an open set U such that $U \subset F(f_n)$ for infinitely many n, then $U \subset F(f)$.*

3 Proof of Theorem 1

It is clear that $\{f_n\}$ converges to f locally uniformly in $\mathbb{C}$.

First, we see $f(z)$. It is a logarithmic lift of

$$h(z) = e^2 z^2 e^{-2z}.$$

The singularity of the inverse are 0 and 1, which are critical points. Both of them are fixed points and thus superattracting fixed points. Since g belongs to the Speiser class, every component of $F(h)$ is simply connected and is eventually mapped on a superattracting component which contains 0 or 1. The logarithmic lift of the superattracting component of 0 is a Baker domain of $F(f)$. The logarithmic lift of the critical point 1 is $\{2k\pi i\}_{k\in\mathbb{Z}}$, each of which is a critical point of f. Since $f(0) = 0$, 0 is a superattracting fixed point. On the other hand, $f(2k\pi i) = 4k\pi i$ implies every Fatou component containing $2k\pi i$ ($k \in \mathbb{Z} \setminus \{0\}$) is wandering. Hence every component of the logarithmic lift of the basin of the superattracting fixed point 1 is eventually mapped on the superattracting component of 0 or is a wandering domain which is eventually mapped on the component containing $2k\pi i$ for some $k \in \mathbb{Z} \setminus \{0\}$.

Next, we see f_n. It is conjugate to

$$g_n(z) = -\frac{1}{2^{n-1} a_n^{a_n+1}} z^{a_n+2} + \frac{2^{n-1}+1}{2^{n-1} a_n} z^2$$

by the translation $z \mapsto z - a_n$, where $a_n = 2^n$. Since

$$g_n'(z) = -\frac{2^{n-1}+1}{2^{n-2} a_n} z \left\{ \left(\frac{z}{a_n} \right)^{a_n} - 1 \right\},$$

the set of the critical points of g_n is $\{0\} \cup \{a_n \exp(ik2\pi/a_n)\}_{k=0}^{a_n-1}$. The origin 0 and 2^n are superattracting fixed points. Elementary calculation shows that every critical point $a_n \exp(ik2\pi/a_n)$ is mapped to a critical point and is eventually mapped to the critical point 2^n. We denote the disc with a center a and a radius r by $D(a,\ r)$. Since, for $w \in \mathbb{C}$,

$$\frac{2^{n-1}+1}{2^{n-2} a_n}(a_n \exp(ik2\pi/a_n) + w) = \left(2 + \frac{1}{2^{n-2}}\right)\left(\exp(ik2\pi/a_n) + \frac{w}{a_n}\right)$$

and

$$\left(\frac{a_n \exp(ik\pi/a_n) + w}{a_n}\right)^{a_n} = \left(1 + \frac{w \exp(-ik\pi/a_n)}{a_n}\right)^{a_n},$$

there exist an N and $r_1 > 0$ such that $D(a_n \exp(ik\pi/a_n),\ r_1)$ is contained in $F(g_n)$ for all $n \geq N$ and for all k with $0 \leq k \leq 2^n - 1$. The set of the critical points of f_n is

$$\{-a_n\} \cup \{-a_n + a_n \exp(ik2\pi/a_n)\}_{k=0}^{a_n - 1}.$$

For fixed k, we have

$$\lim_{n\to\infty} -a_n + a_n \exp(ik2\pi/a_n) = i2k\pi.$$

Furthermore, since the component of $F(f)$ containing $2k\pi i$ is a logarithmic lift of the component of $F(f)$ containing 1, every component is given by

$$\{z + 2k\pi i \mid z \in A\},$$

where A is a component of $F(f)$ containing 0. Hence there exists an r_2 such that $D(2k\pi i,\ r_2)$ is contained in $F(f)$ for all $k \in \mathbb{Z}$. Let $r = \min(r_1, r_2)$ then, for fixed k, $\{D(-a_n + a_n \exp(ik2\pi/a_n),\ r)\}$ converges to $D(i2k\pi,\ r)$ in the Carathéodory sense. To see the Carathéodory convergence of Fatou sets, we show that, for an arbitrary compact set K contained in a component of $F(f)$, there exists an N such that it is contained in $F(f_n)$ for all $n \geq N$ from Lemma 4. Take a Fatou component D_0 containing $c_0 = 2k\pi i$ for some $k \in \mathbb{Z} \setminus \{0\}$. Let K be a compact set in D_0. We set $D_n = f^n(D_0)$. Every D_n contains the critical point $c_n = 2^n k\pi i$ and $f^n(c_0) = c_n$. Since D_n is simply connected, there exists a conformal map from D_n onto the unit disk satisfying $\varphi_n(c_n) = 0$ so that

$$\varphi_n \circ f^n \circ \varphi_0^{-1}(z) = z^n$$

on D_0. Since

$$D_n = \{z + 2k(2^{n-1} - 1)\pi i \mid z \in D_0\},$$

φ_n can be given by

$$\varphi_n(z) = \varphi_0(z - 2k(2^{n-1} - 1)\pi i).$$

Hence there exist an N and M such that

$$f^N(K) \subset D(2^N k\pi i,\ r/2) \subset D(-a_n + a_n \exp(ik2\pi/a_n),\ r),$$

for all $n \geq M$. Because of the backward invariance of Fatou sets, we have $K \subset F(f_n)$ for all $n \geq M$. It follows that D_0 is the limit in the Carathéodory sense. Since every wandering domain is eventually mapped on a component containing a critical point, we see that it is the limit in the Carathéodory sense by the argument above.

The argument in [6] shows that the components of the attracting basin of the superattracting fixed point 0 of f are the limits in the Carathéodory sense. The argument in [9] shows that the components which are eventually mapped on the Baker domain are the limits in the Carathéodory sense. Thus we obtain that $\{F(f_n)\}$ converges to $F(f)$ in the Carathéodory sense, which implies $\{J(f_n)\}$ converges to $J(f)$ in the Hausdorff metric. ∎

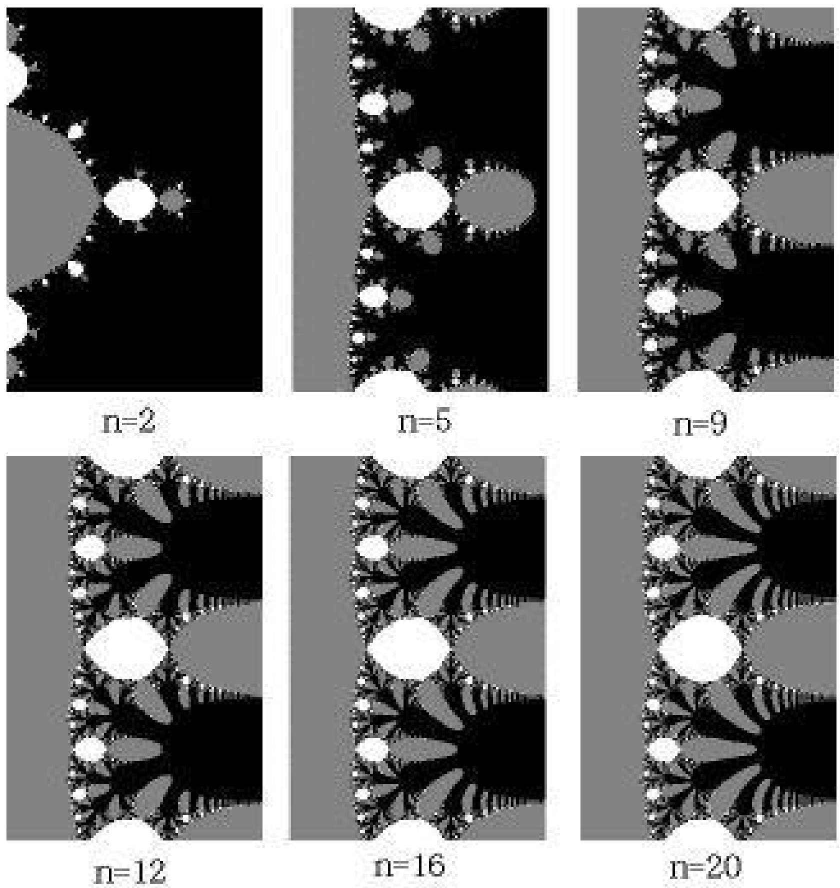

Figure 2: The Fatou sets of the approximating polynomials f_n. The basin of the attraction of $-a_n$ is grey. The basin of the attraction of 0 is white. The basin of the attraction at infinity is black.

4 Remarks

In [9], we considered $f(z) = z - e^z + 1 + 2\pi i$, whose Fatou set consists only of wandering domains. This function was first given in [1]. We take

$$f_n(z) = e^{i\frac{2\pi}{n}} \left(1 + \frac{1}{n}\right) z - e^{i\frac{2\pi}{n}} \left(1 + \frac{z}{n}\right)^{n+1} + n \left\{ e^{i\frac{2\pi}{n}} \left(1 + \frac{1}{n}\right) - 1 \right\}$$

as a locally uniformly convergent sequence to f. Each f_n has a superattracting cycle with the period n. We denote the wandering domain containing a critical point 0 by D and the attracting component containing a critical point 0 of f_n by D_n. In [9], we showed that $\{D_n\}$ converges to D in the Carathéodory sense. Each D_n comes back to itself after n-th iterate of f_n. Roughly speaking, as the sequence of period tends to infinity, the cyclic component finally becomes wandering. In the case of the function we considered in this note, the period of each attracting cycle of the polynomials is 1. We denote the wandering domain of f containing $2\pi i$ by D and the component of f_n containing $-2^n + 2^n \exp(i\pi/2^n)$ by D_n. After n-th iterate of f_n, D_n is mapped to the attracting component containing 0. We have already seen that $\{D_n\}$ converges to D in the Carathéodory sense. Roughly speaking again, as the length of the iterate to the superattracting component tends to infinity, the component finally becomes wandering. In both cases, the limit function of the wandering domain is infinite. Eremenko and Lyubich [5] gave an example of a transcendental entire function which has a wandering domain whose limit functions are finite. Since they constructed it by using Arakeljan's approximation theorem, the concrete formula is not known. So we give the following question.

Question 1 *Can one give an example of entire function which has a wandering domain whose limit function are finite by using a sequence of polynomials converging locally uniformly in* $\mathbb{C}$ *?*

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