100 THE CARATHÉODORY CONVERGENCE OF FATOU COMPONENTS OF POLYNOMIALS TO BAKER DOMAINS OR WANDERING DOMAINS [1]

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1 Introduction

Let f be a rational function or a transcendental entire function. Consider X is a complex sphere $\widehat{\mathbb{C}}$ if f is a rational function and X is a complex plain $\mathbb{C}$ if f is a transcendental entire function. The maximal open subset of X on which the family $\{f^n\}$ is normal is called the Fatou set of f and we denote it by $F(f)$. The complement of $F(f)$ in $\widehat{\mathbb{C}}$ is called the Julia set of f and we denote it by $J(f)$. We remark that, in this note, ∞ is always contained in the Julia sets of transcendental entire functions. Fatou sets and Julia sets are completely invariant. Hence a component of $F(f)$ is mapped into some component of $F(f)$ by f. We

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say that a component of D of $F(f)$ is a cyclic component if $f^p(D)$ is contained in D for some natural number p. If $p = 1$, then it is called invariant. All the kinds of cyclic components are already known. They are an attracting component, a parabolic component, a Siege disk, a Herman ring and a Baker domain. The first three have relationship with cyclic points and the first four have relationship with singular values. On the other hand, Baker domains have relationship with neither cyclic point nor singular value. Baker domains are never appeared in the case of rational functions. Some facts on Baker domains are stated in § 3. We say that a component D is a wandering domain if $f^n(D) \neq f^m(D)$ for all natural numbers n and m with $n \neq m$. Due to Sullivan's theorem, we know that there exists no wandering domain in the case of rational functions. On the contrary, in the case of transcendental entire functions, there may exist wandering domains if it has infinitely many singular values.

Let f be a transcendental entire functions and $\{f_n\}_{n=1}^{\infty}$ be a sequence of polynomials converging to f locally uniformly on $\mathbb{C}$. Such sequences always exist. For example, the Taylor series of f is so. How about the convergence of the dynamics of f_n to that of f? Devaney, Goldberg and Hubbard ([3]) considered the exponential family $E_\lambda(z) = \lambda e^z$ and polynomials $P_{\lambda,n}(z) = \lambda[1 + (z/n)]^n$ as a convergent sequence and showed the convergence of hyperbolic components and external rays in the parameter space. After this result, Krauskopf([9]) showed that if E_λ has an attracting component, then $J(P_{\lambda,n})$ converges to $J(E_\lambda)$ in the Hausdorff metric. The definition of convergence in the Hausdorff metric is given in § 2. Further, Kisaka([8]) proved more general results.

Theorem 1 *Let f be a transcendental entire function and $\{f_n\}_{n=1}^{\infty}$ be a sequence of polynomials converging to f locally uniformly on $\mathbb{C}$. If $F(f)$ consists only of basins of attracting cycles, then $J(f_n)$ converges to $J(f)$ in the Hausdorff metric.*

The key fact for the results of Krauskopf and Kisaka is that an arbitrary cycle of f is approximated by some cycle of f_n, which immediately follows from the Hurwitz theorem.

In § 4, we consider examples of transcendental entire functions whose Fatou sets have Baker domains or wandering domains and whose Julia sets are approximated by Julia sets of polynomials.

2 Hausdorff convergence and Carathéodory convergence

First, we introduce an idea of convergence of Julia sets. Let ρ be the chordal metric on $\widehat{\mathbb{C}}$. The ϵ-neighborhood of a set A is denoted by $U_\epsilon(A)$. Let A and B be nonempty compact sets in $\widehat{\mathbb{C}}$. The Hausdorff distance $d(A,\ B)$ of A and B is defined by

$$d(A,\ B) = \inf\{\varepsilon > 0 \mid A \subset U_\varepsilon(B), \quad B \subset U_\varepsilon(A)\}.$$

This distance defines the Hausdorff metric on the set of all the non-empty compact sets in $\widehat{\mathbb{C}}$. Let K and K_n $(n \in \mathbb{N})$ be non-empty compact sets in $\widehat{\mathbb{C}}$. We say that K_n converges to K in the Hausdorff metric if $d(K_n,\ K) \to 0$ as $n \to \infty$. That is, for an arbitrary $\epsilon > 0$, there exists an N such that

$$K \subset U_\varepsilon(K_n) \quad \text{and} \quad K_n \subset U_\varepsilon(K)$$

for all $n \geq N$. We denote it by

$$K_n \to K \qquad \text{(Hausdorff metric).}$$

Since the set of all the cyclic points are dense in the Julia set, it is easy to see the following.

Proposition 2 *If f_n converges to f locally uniformly, then, for an arbitrary $\epsilon > 0$, there exists an N such that*

$$J(f) \subset U_\varepsilon(J(f_n))$$

for all $n \geq N$.

This proposition implies the Hausdorff convergence of Julia sets is *lower semicontinuous* (see Douady [4]).

Next, we introduce an idea of convergence of open sets. Let O and O_n $(n \in \mathbb{N})$ be open sets in $\widehat{\mathbb{C}}$. We say that O_n converges to O in the Carathéodory sense if the following two conditions hold:
(1) for an arbitrary compact set $I \subset O$, there exits an N such that $I \subset O_n$ for all $n \geq N$ and
(2) if an open set U is contained in O_m for infinitely many m, then $U \subset O$.
We denote the convergence of O_n to O in the Carathéodory sense by

$$O_n \to O \qquad \text{(Carathéodory sense).}$$

Two ideas of convergence defined above have the following relationship.

Lemma 3 *Non-empty closed set K_n converges to K in the Hausdorff metric if and only if the complement K_n^c of K_n converges to the complement K^c of K in the Carathéodory sense.*

Proposition 2 is rephrased as follows.

Lemma 4 *Assume f_n converges to f locally uniformly. If there exists an open set U such that $U \subset F(f_n)$ for infinitely many n, then $U \subset F(f)$.*

3 Baker domains

Let f be a transcendental entire function. An invariant component B of $F(f)$ is called a Baker domain if the limit function of $\{f^n\}$ is only ∞. Hence a Baker domains have no fixed point in it. Eremenko and Lyubich ([6]) showed that, if a transcendental entire function has a Baker domain, then the set of the singular values is unbounded. They ([5]) also gave an example of a Baker domain which has no singular value in it. Herman ([7]) also gave an example of a Baker domain having a property which the example given by Eremenko and Lyubich has and further its boundary is contained in the closure of the forward orbits of singular values. Moreover, Bergweiler ([2]) gave an example of a Baker domain such that the Euclidean distance between it and the set of all the singular values is positive.

A subdomain V of a Baker domain B is called a fundamental set of B, if V is simply connected, $f(V) \subset V$ and, for every compact set $K \subset B$, $f^n(K) \subset V$ for some n.

4 Examples

Example 1 *Let $f(z) = z + e^z - 1$. Then $F(f)$ has a Baker domain. Take*

$$f_n(z) = \left(1 - \frac{1}{n}\right) z + \frac{n-1}{n+1}\left(1 + \frac{z}{n}\right)^{n+1} - 1$$

as a locally uniformly convergent sequence to f. Then we have

$$J(f_n) \to J(f) \qquad \text{(Hausdorff metric)}.$$

Take

$$g_n(z) = z + \left(1 + \frac{z}{2n}\right)^{2n} - 1$$

as a locally uniformly convergent sequence to f. Then we have

$$J(g_n) \not\to J(f) \qquad \text{(Hausdorff metric)}.$$

First, we consider a transcendental entire function $h(z) = ze^{z-1}$. It has only two singular values 0 and $h(-1)$. Hence $F(h)$ has neither a wandering domain nor a Baker domain. It also has an attracting fixed point 0. The inverse image of sufficiently small open disk with center 0 consists of two components. Thus the attracting component of 0 contains -1 and $F(h)$ consists only of this component. The function $f(z)$ is a logarithmic lift of $h(z)$. Therefore $F(f)$ consists of only one Baker domain B, which is a lift of the attracting component of $F(f)$ (see [7], [1] and [10]). See Figure 1.

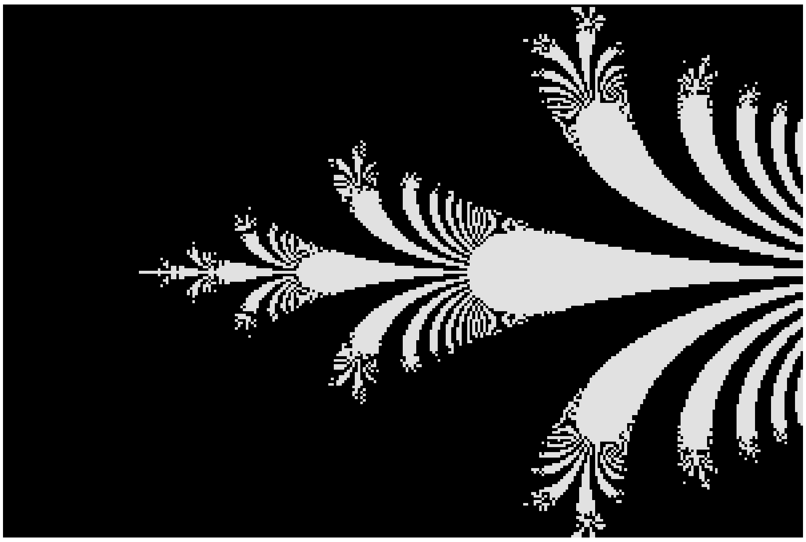

Figure 1: Julia set of $f(z) = z + e^z - 1$

Since

$$f_n'(z) = 1 - \frac{1}{n} + \frac{n-1}{n}\left(1 + \frac{z}{n}\right)^n,$$

we see that $-n$ is an attracting fixed point and finite critical points are

$$z = -n + n\exp(1+2k)\frac{\pi}{n}i \qquad (0 \leq k \leq n-1).$$

We denote the attracting component containing $-n$ by A_n. Since A_n contains at least one critical point and $f_n(z)$ is symmetric with respect to a rotation with center $-n$, all the finite critical points are contained in A_n. Hence $F(f_n)$ consists of A_n and the superattracting component of the point at infinity. The boundary of A_n is a simple closed curve.

Set

$$U_n = \{z \mid |z+n| < n\}.$$

Then we have $f_n(U_n) \subset U_n$ and hence $U_n \subset A_n$. From Lemma 4,

$$\bigcap_{n\geq 1} \bigcup_{k\geq n} U_k = \{z \mid \text{Re } z < 0\}$$

is contained in $F(f)$ and contains a fundamental set of B. Hence we have

$$A_n \to B \qquad \text{(Carathéodory sense)}.$$

By Lemma 3, we have

$$A_n^c \to B^c = J(f) \qquad \text{(Hausdorff metric)}.$$

Since $J(f)$ has no interior point, we obtain

$$J(f_n) \to J(f) \qquad \text{(Hausdorff metric)}.$$

See Figure 2.

Next, we consider the sequence $\{g_n\}$. The fixed points of g_n are

$$z_n^{(k)} = -2n + 2n \exp \frac{k}{n}\pi i \qquad (0 \le k \le 2n-1).$$

Since

$$g_n'(z) = 1 + \left(1 + \frac{z}{2n}\right)^{2n-1},$$

we have the critical points

$$w_n^{(k)} = -2n + 2n \exp \frac{2k+1}{2n-1}\pi i \qquad (0 \le k \le 2n-2)$$

and we see that, for $2n/3 < k < 4n/3$, $z_n^{(k)}$ are attracting fixed points of g_n. Since each attracting component has a fixed point on its boundary. The origin is a fixed point which is on the boundary of the attracting component of $z_n^{(n)} = -4n$. A closed interval $[-4n,\ 0]$ is a forward invariant curve $L_{n,1}$ under g_n and $L_{n,1} \setminus \{0\}$ is contained in the attracting component of $-4n$. We can also see that $z_n^{(1)} = -2n + 2n\exp(\pi i/n)$ is a fixed point which is on the boundary of the attracting component of $z_n^{(n-1)} = -2n + 2n\exp[(n-1)\pi i/n]$. There exists a forward invariant curve $L_{n,2}$ joining $z_n^{(1)}$ and $z_n^{(n-1)}$ in the strip $\{z \mid \pi \le \text{Im } z \le 3\pi\}$ and $L_{n,2} \setminus z_n^{(1)}$ is contained in the attracting component of $z_n^{(n-1)}$. There exist points of $J(g_n)$ between $L_{n,1}$ and $L_{n,2}$. Hence we have

$$J(g_n) \not\to J(f) \qquad \text{(Hausdorff metric)}.$$

See Figure 3.

Example 2 *Let $f(z) = z - e^z + 1 + 2\pi i$. Then $F(f)$ has wandering domains. Take*

$$f_n(z) = e^{i\frac{2\pi}{n}}\left(1 + \frac{1}{n}\right)z - e^{i\frac{2\pi}{n}}\left(1 + \frac{z}{n}\right)^{n+1} + n\left\{e^{i\frac{2\pi}{n}}\left(1 + \frac{1}{n}\right) - 1\right\}$$

as a locally uniformly convergent sequence to f. Then we have

$$J(f_n) \to J(f) \qquad \text{(Hausdorff metric)}.$$

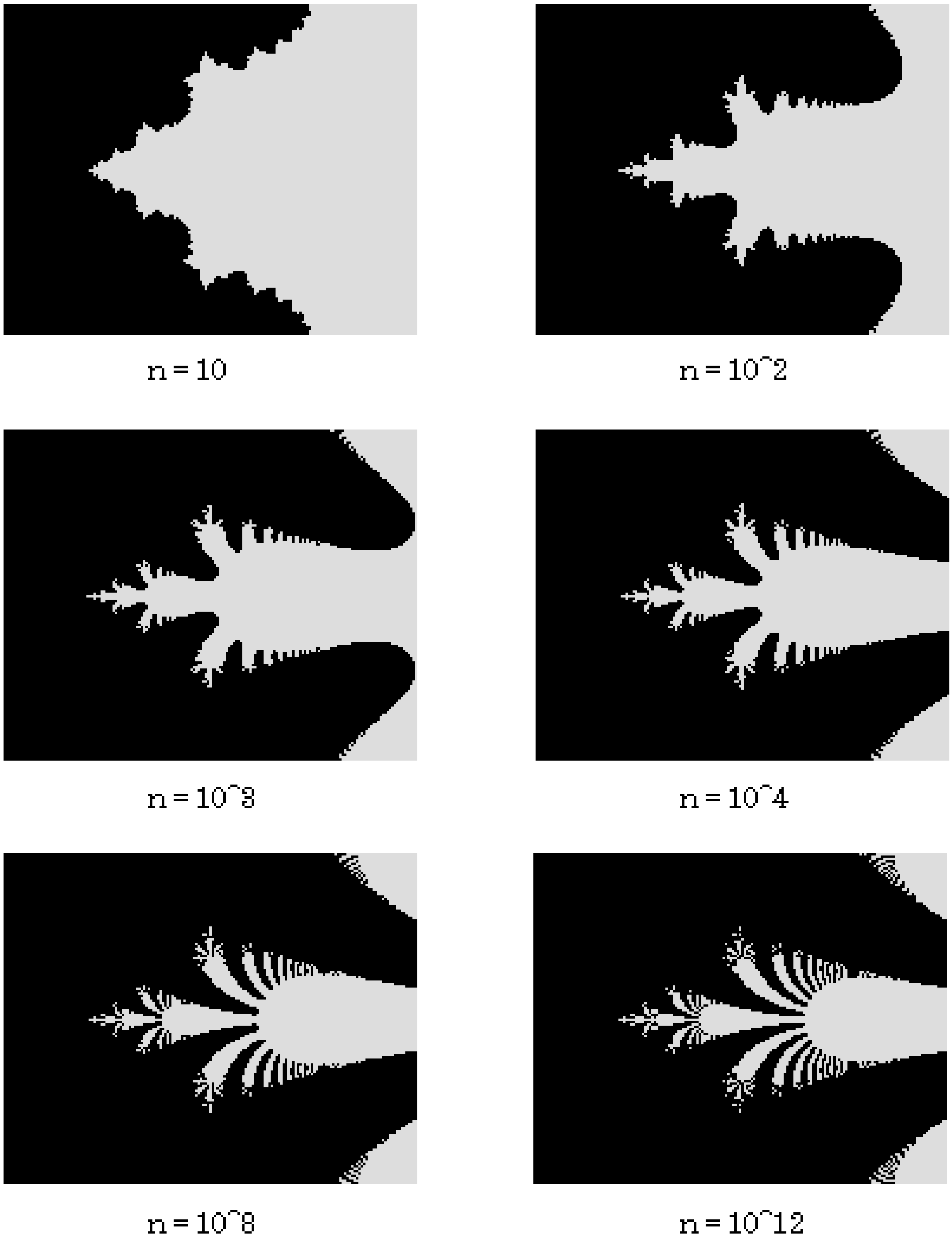

Figure 2: Fatou sets of $f_n(z) = [1-(1/n)]z + [(n-1)/(n+1)][1+(z/n)]^{n+1} - 1$

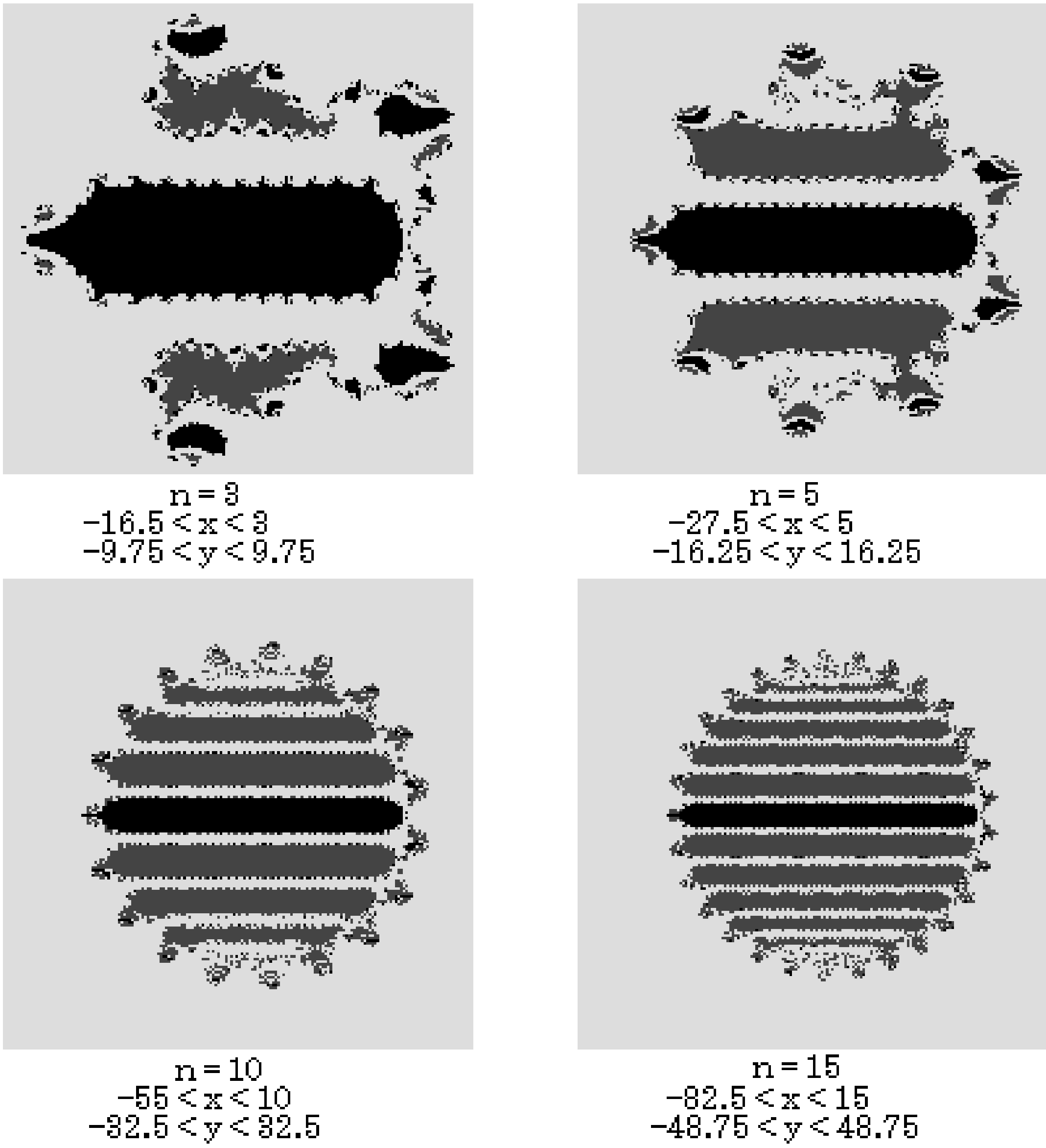

Figure 3: Fatou sets of $g_n = z + [1 + (z/2n)]^{2n} - 1$

First we consider a transcendental entire function $h(z) = z - e^z + 1$. Its singular values are $a_n = 2n\pi i$ $(n \in \mathbb{Z})$, which are also superattracting fixed points. Since $h(z)$ and $f(z)$ satisfy the following two formulae

$$h(z + 2\pi i) = h(z) + 2\pi i \qquad \text{and} \qquad f(z) = h(z) + 2\pi i,$$

we have $J(f) = J(h)$ (see [1] and [10]). Hence the component containing a_n is a wandering domain of f. We take

$$h_n(z) = \left(1 + \frac{1}{n}\right) z - \left(1 + \frac{z}{n}\right)^{n+1} + 1$$

as a locally uniformly convergent sequence to h. Since $F(h)$ consists only of attracting basins, $J(h_n)$ converges to $J(h)$ in the Hausdorff metric by Theorem 1. It is clear that $f_n^n(z) = h_n^n(z)$, which implies $J(f_n) = J(h_n)$. Therefor, we have

$$J(f_n) \to J(f) \qquad\qquad \text{(Hausdorff metric)}.$$

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