Julia sets of subhyperbolic rational functions

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Abstract

In this note, we study the Julia sets of subhyperbolic rational functions. The boundaries of simply connected attracting components of the Fatou sets of such functions are closed curves. We show that, under certain conditions, they are Jordan curves. Making use of this result, we see that the Fatou set of Newton's method for $z^3 - 1 = 0$ consists of Jordan domains.

1 Introduction

Let f be a rational function of degree at least two. We denote by $F(f)$ and $J(f)$ the Fatou set and the Julia set of f, respectively. In this note, we study the Julia sets of subhyperbolic rational functions, whose definition is stated in § 2.

In § 2, we give two theorems on completely invariant components and the residual Julia sets which were proved in [8] for hyperbolic rational functions.

The invariant Fatou components of subhyperbolic rational functions are only attracting ones. Moreover, in this case, the boundary of a simply connected Fatou component is a closed curve. However, it is not necessary a Jordan curve. In § 3, we study some properties of such boundaries as subsets of the Julia set.

In § 4, we show that the boundary of each Fatou component of some subhyperbolic rational functions is a Jordan curve. For example, the rational function $f(z) = (2z^3 + 1)/3z^2$, which is well-known as Newton's method for $z^3 - 1 = 0$, satisfies the condition of our theorem. Hence $F(f)$ consists of Jordan domains. The figure of its Julia set is found, for example, in [9].

In § 5, we give an example of a rational function whose Julia set is a Sierpinski carpet.

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2 Subhyperbolic rational functions

A point ζ is a critical point of a rational function f if f fails to be injective in any neighborhood of ζ. We say that a rational function f is subhyperbolic, if
(1) every critical orbit in $J(f)$ is eventually periodic, and
(2) every critical orbit in $F(f)$ is attracted to an attracting cycle.

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It is equivalent to that there exists some admissible metric on some neighborhood of $J(f)$ such that f is expanding on $J(f)$.

Let ζ be a critical point of f. Choose some Möbius maps u and v such that $u^{-1}(\zeta)$ and $vf(\zeta)$ is in $\mathbf{C}$. We say that ζ is a critical point with multiplicity t if $(vfu)'(z)$ has t zeros at $u^{-1}(\zeta)$. We denote by S^1 the unit circle. The first half of the following lemma is found in [2, p.94].

Lemma 1 *Let f be a subhyperbolic rational function. Assume that $F(f)$ has a simply connected attracting component D, on which f has a local degree k. Then there exists a continuous function $\varphi : S^1 \to \partial D$ such that*

$$f \circ \varphi(z) = \varphi \circ h(z),$$

where $h(z) = z^k$. Moreover, unless ζ is a critical point of f, then h is injective on $\varphi^{-1}(\zeta)$. If ζ is a critical point with multiplicity t, then h is at most $t+1$ to 1 on $\varphi^{-1}(\zeta)$.

Proof. Since $f|_D$ is conformally conjugate to a finite Blaschke product of degree k, there exist a compact set E of D, $A = \{z \mid r < |z| < 1\}$ and a homeomorphism φ from A onto $D \setminus E$ such that

$$f \circ \varphi(z) = \varphi(z^k)$$

for $z \in \{z \mid r^{1/k} < |z| < 1\}$. For the sake of expandingness of f, the formula above is valid for $z \in S^1$. We omit the detail, which can be found in [2, p.94].

Defining $e(z) = \{lz \mid r^{1/k} < l \leq 1\}$ for $z \in S^1$, we have

$$f \circ \varphi(e(z)) = \varphi \circ h(e(z)) = \varphi(e(z^k)).$$

Take $\zeta \in \partial D$ and $z, w \in \varphi^{-1}(\zeta)$ $(z \neq w)$. Supposing $h(z) = h(w)$, we have

$$\varphi(e(z)) \cap \varphi(e(w)) = \{\zeta\} \quad \text{and} \quad f \circ \varphi(e(z)) = f \circ \varphi(e(w)).$$

Hence ζ is a critical point of f. If ζ is a critical point with multiplicity t, then it is clear that f is at most $t+1$ to 1 on φ^{-1}. □

The first half of Lemma 1 shows that the boundary of a simply connected component is locally connected. Furthermore, it was shown in [4] that if the Julia set of a rational function belonging to a certain class, which contains all the subhyperbolic rational functions, is connected, then it is locally connected.

In [8], we define the residual Julia set, which is the set of those points in the Julia set not lying in the boundary of any component of the Fatou set. We denote it by $J_0(f)$ for a rational function f. Using Lemma 1, we deduce the following theorems by the argument similar to that in [8].

Theorem 2 *Let f be a subhyperbolic rational function with degree at least two. Assume that $J(f)$ is connected. If there exists a forward invariant component D of $F(f)$ with $\partial D = J(f)$, then D is completely invariant.*

Theorem 3 *Let f be a subhyperbolic rational function with degree at least two. Then $J_0(f)$ is empty if and only if $F(f)$ has a completely invariant component or consists of only two component.*

For proofs of the theorems above, see [8].

3 Boundaries of attracting components

If the boundary of a simply connected attracting component is not a Jordan curve, then $\varphi^{-1}(z)$ consists of more than two points for some z in the boundary. For a point z in the boundary, we say that it is simple if $\varphi^{-1}(z)$ consists of one point. We have the following.

Lemma 4 *Let f be a subhyperbolic rational function, D a simply connected component of $F(f)$ and φ a function defined in Lemma 1. Then there exists N such that the cardinal number of $\varphi^{-1}(\zeta)$ is at most N for all ζ in ∂D.*

Proof. Let C be the set of the critical points on ∂D. We set

$$[C] = \{z \mid f^n(z) = f^m(\zeta) \quad \text{for some } \zeta \in C \quad \text{and} \quad n, m \in \mathbf{N} \cup \{0\}\}$$

and $C' = [C] \cap \partial D$. Subhyperbolicity of f implies that C' contains only finite many cyclic points such that an arbitrary point of C' is eventually mapped in them. In [6], it was shown that, for a cyclic point ζ, the cardinal number of $\varphi^{-1}(\zeta)$ is finite. By Lemma 1, there exists N' such that the cardinal number of $\varphi^{-1}(\zeta)$ is at most N' for all $\zeta \in C'$.

For $x, y \in S^1$, we denote by $[x, y]$ the closed arc on S^1 whose end points are x and y and which is contained in a semicircle and by $d(x, y)$ the arc length of $[x, y]$.

We claim that, for sufficient small $\epsilon > 0$, there exists δ such that, for an arbitrary $\zeta \in \partial D \setminus C'$, if $x,\ y \in \varphi^{-1}(\zeta)$ with $[h(x), h(y)] \cap h\varphi^{-1}(C) = \emptyset$ satisfy $d(h(x), h(y)) < \delta$, then $d(x, y) < \epsilon$. Assume the claim were false. Then there exist $\epsilon > 0$ and $\{\zeta_n\} \subset \partial D$ such that $x_n,\ y_n \in \varphi^{-1}(\zeta_n)$ with $[h(x_n), h(y_n)] \cap h\varphi^{-1}(C) = \emptyset$ satisfy

$$d(x_n, y_n) \geq \epsilon$$

for all $n \in \mathbf{N}$ and

$$d(h(x_n), h(y_n)) \to 0$$

as $n \to \infty$. If necessary, taking subsequences, we may assume

$$\zeta_n \to \zeta, \qquad x_n \to x \quad \text{and} \quad y_n \to y$$

for some ζ, x and y as $n \to \infty$. These show that

$$x,\ y \in \varphi^{-1}(\zeta), \qquad x \neq y \quad \text{and} \qquad h(x) = h(y).$$

From Lemma 1, ζ is a critical point of f. Let l and l_n be simple curves in A whose end points are x and y and x_n and y_n, respectively. Then $\varphi(l) \cup \varphi(x)$ and $\varphi(l_n) \cup \varphi(x_n)$ are Jordan curves in $\hat{\mathbf{C}}$. If only one of x_n and y_n is in $[x, y]$, then we can choose l and l_n which cross each other at exactly one point. Hence $\varphi(l) \cup \varphi(x)$ and $\varphi(l_n) \cup \varphi(x_n)$ cross each other at exactly one point, because of $\varphi(x) \neq \varphi(x_n)$. It is impossible. Thus we have $x_n,\ y_n \in [x, y]$ or $x_n,\ y_n \in [x, y]^c$. Since h is a proper covering map on S^1 and $h(x_n) \neq h(y_n)$, we have

$$[h(x_n), h(y_n)] \cap h(x) \neq \emptyset,$$

for sufficiently large n. This contradicts the assumption.

For a set U in S^1, we denote by diam U the diameter of U. There exists a finite open covering $\{U'_i\}_{i=1}^{M'}$ of S^1 which satisfies diam $U'_i < \delta$. Since f is subhyperbolic, $C_f = \cup_{n=0}^{\infty} f^n(C)$ is a finite set. It follows that $U'_i \backslash C_f$ consists of finite open sets. Gathering such open sets, we construct an open covering $\{U_i\}_{i=1}^{M}$ of $S^1 \backslash C_f$. Obviously, we have diam $U_i < \delta$. For an arbitrary $\zeta \in \partial D \backslash C'$, let

$$X_n = \varphi^{-1}(f^n(\zeta)) = h^n(\varphi^{-1}(\zeta)).$$

We restrict U_i to X_n and denote it by U_i again. If δ is sufficiently small, then the function $h(z) = z^k$ implies

$$\text{diam } h^{-1}(U_i) = \frac{1}{k}\text{diam } U_i \leq \frac{1}{k}\delta.$$

Setting $U_i^{(1)} = h^{-1}(U_i)$, we have

$$X_{n-1} = U_1^{(1)} \cup \cdots \cup U_M^{(1)}.$$

Repeating this procedure, we obtain

$$X_0 = U_1^{(n)} \cup \cdots \cup U_M^{(n)},$$

where diam $U_i^{(n)} \leq \delta/k^n$. We obtain the cardinal number of X_0 is at most M as n is arbitrary. Take $N = \max\{N', \ M\}$, which we require. □

Concerning the following theorem, Przytycki and Zdunik ([11]) have already proved it for the set A_1 in the following under more generous assumption.

Theorem 5 *Let f be a subhyperbolic rational function and D a simply connected component of $F(f)$. Then each of the following three sets is dense in ∂D.*

$$\begin{aligned}
A_1 &= \{\zeta \in \partial D \mid \zeta \text{ is a cyclic point of } f\} \\
A_2 &= \{\zeta \in \partial D \mid \{f^n(\zeta)\}_{n=0}^{\infty} \text{ is dense in } \partial D\} \\
A_3 &= \{\zeta \in \partial D \setminus \cup_{n=0}^{\infty} f^{-n}(A_1) \mid \{f^n(\zeta)\}_{n=0}^{\infty} \text{ is not dense in } \partial D\}
\end{aligned}$$

Proof. For $z = e^{2\pi i t}$ $(0 \leq t < 1)$, we write t by a base k representation

$$t = \sum_{j=1}^{\infty} a_j k^{-j} \qquad\qquad (a_j = 0, 1, \cdots, k-1),$$

where we assume that infinitely many a_j's are not 0. The map ϕ from S^1 to the sequence space on k symbols is defined by

$$\phi(z) = (a_1 a_2 \cdots).$$

Denoting by σ the shift map on Σ_k, we have

$$\phi \circ h = \sigma \circ \phi.$$

Let A'_1, A'_2 and A'_3 be the subsets of Σ_k as follows,

$$\begin{aligned} A'_1 &= \{u \in \Sigma_k \mid u \text{ is cyclic}\} \\ A'_2 &= \{u \in \Sigma_k \mid u \text{ contains all the finite sequences of } k \text{ symbols}\} \\ A'_3 &= \{u \in \Sigma_k \setminus \cup_{n=0}^{\infty} \sigma^{-n}(A'_1) \mid u \text{ does not contains a certain finite sequence of } k \text{ symbols}\} \end{aligned}$$

It is easy to see that $\phi^{-1}(A'_i)$ is dense in S^1 for each i. Using σ and Lemma 4, we have

$$\varphi \circ \phi^{-1}(A'_i) = A_i \qquad (i = 1,\ 2,\ 3)$$

Thus, by Lemma 4, A_i is dense in ∂D for each i. □

4 Fatou set consisting of Jordan domains

In [10], Pilgrim showed that the existence of a Fatou component with Jordan curve boundary for a certain critically finite hyperbolic rational functions.

We give a fundamental lemma for Jordan domains in Fatou sets.

Lemma 6 *Let f be a subhyperbolic rational function and B a forward invariant component of $F(f)$. If there exist a complementary component E of $\overline{B}$ and a component of D of $F(f)$ such that $E \supset D \cup f^{-1}(D)$, then the boundary of B is a Jordan curve.*

Proof. Let U be a complementary component of $\overline{B}$ such that $U \neq E$. If $f(U) \cap E \neq \emptyset$, then we have $f(U) \supset E$. This means that there exists a component D' of $F(f)$ in U such that $f(D') = D$. This contradicts the assumption. Hence we have $f^{-1}(E) \subset E$. From a fundamental property of the Julia set (see [1, p.71]), for an arbitrary $\zeta \in E$ but at most two points, we have

$$J(f) \subset \overline{\cup_{n=0}^{\infty} f^{-n}(\zeta)} \subset \overline{E}.$$

Thus B is simply connected and its boundary is a Jordan curve. □

Theorem 7 *Let f be a subhyperbolic rational function with degree three. Suppose that f has three attracting fixed points and that there exists no completely invariant component. Then $F(f)$ consists of Jordan domains.*

Proof. Let D_i $(i = 1,\ 2,\ 3)$ be attracting components of $F(f)$. Since the degree of f is three, there exists only one repelling fixed point ζ of f. Shishikura ([12]) showed that if the rational function has only one fixed point which is repelling or parabolic with multiplier 1, then its Julia set is connected. Hence $J(f)$ is connected.

First we show that the boundary of D_i is a Jordan curve. Then it is clear that so is the boundary of each component of $f^{-n}(D_i)$ for all $n \in \mathbf{N}$. From Lemma 1, the repelling fixed point ζ is on each boundary of D_i. It was shown in [3] that, for a fixed point ζ on ∂D, $\varphi^{-1}(\zeta)$ consists of periodic points of h with same cycles. Let φ_i be the function defined in Lemma 1 for D_i. Since D_i is not completely invariant, the local degree of f on D_i is two and $h(z) = z^2$. It follows that $\varphi_i^{-1}(\zeta)$ consists of only one point, that is, $1 \in S^1$. Thus ζ is a simple point with respect to ∂D_i. Further, the fact $h^{-1}\varphi_i^{-1}(\zeta) = \{1, -1\}$ implies that $\varphi_i(-1) = \zeta_i \in \partial D_i$ is simple with respect to ∂D_i.

Assume that $\eta = \zeta_1 = \zeta_2 = \zeta_3$. Choose a simple curve l_i in D_i whose end points are ζ and ζ_i. Unless η is a critical point, then f is an orientation preserving homeomorphism near η. That is, the cyclic order of l_1, l_2 and l_3 near η is preserved by f. This is impossible. Thus η is a critical point and $f^{-1}(\zeta) = \{\zeta, \eta\}$. There exists only one component D_i' of $f^{-1}(D_i)$ which is different from D_i. Then $\partial D_i'$ contains η. Hence the complementary component of $\overline{D_i}$ whose boundary contains η contains D_j and D_j' for $j \neq i$ (see Figure 1, where circles are attracting components and squares are their inverse images). From Lemma 6, the boundary of each D_j is a Jordan curve.

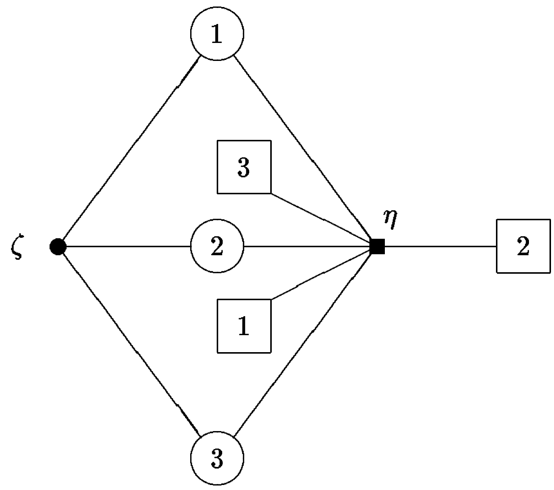

Figure 1:

Since degree of f is three, $f^{-1}(\zeta)$ consists at most three points and $f^{-1}(\zeta) \ni \zeta$. Now we may assume $\zeta_1 \neq \zeta_2 = \zeta_3$ and set $\eta = \zeta_2 = \zeta_3$. We also assume that D_1 contains ∞. Let l_2 and l_3 be curves defined above. Then $l_2 \cup l_3 \cup \{\zeta, \eta\}$ is a Jordan curve and is denoted by γ. We also denote by A_1 the complementary component of γ which contains ∞ and by A_2 the other component. The point ζ_1 is in both $\partial D_2 \cup \partial D_2'$ and $\partial D_3 \cup \partial D_3'$. Since $\zeta_1 \neq \eta$, we have $\zeta_1 \in \partial D_2'$, $\zeta_1 \notin \partial D_2$, $\zeta_1 \in \partial D_3'$ and $\zeta_1 \notin \partial D_3$. Hence D_2' and D_3' are contained in A_1 as so is ζ_1. For $i = 2,\ 3$, the boundary of D_i contains two inverse images of η for f, one of which is in A_1 and the other in A_2. The number of the inverse images of η is at most three. Since D_2' and D_3' are in A_1, there exists only one inverse image of η in A_2. Moreover, $\partial D_2 \cap \partial D_3'$ and $\partial D_3 \cap D_2'$ contain inverse images of η. Because f is an orientation preserving homeomorphism, D_1' must be contained in A_2 (see Figure 2, where $\blacksquare$'s are inverse images of η).

Note that the inverse images of η are not contained in ∂D_1. For D_1, D_2 and D_2' are contained

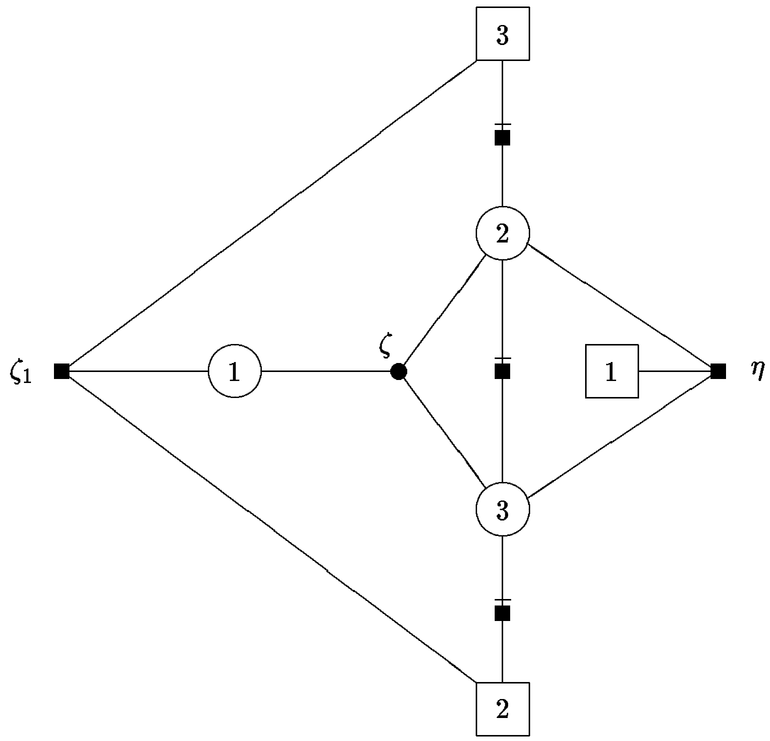

Figure 2:

in the complementary component of $\overline{D_1}$ whose boundary contains ζ. By Lemma 6, ∂D_1 is a Jordan curve. For D_2, D_1 and D_3 are contained in the complementary component of $\overline{D_2}$ whose boundary contains ζ. Since ζ_1 is not in ∂D_2, D_3' is also contained in it. Thus, by Lemma 6, ∂D_2 is a Jordan curve. By using the same argument, ∂D_3 is a Jordan curve.

Next, $F(f)$ may have an attracting cyclic component with period p (≥ 2). Since such cyclic components contain a critical point, it occurs in only the case in Figure 2. Let $B_1,\ B_2, \cdots, B_p$ be the cyclic components. If ∂B_i contains a point of $f^{-n}(\zeta)$ for some n, then it is clear that ζ is in ∂B_i and hence is in ∂B_j for all j ($1 \leq j \leq p$). Then f maps B_j to each other. On the other hand, f preserves each D_i for $i = 1,\ 2,\ 3$. It contradicts that f is an orientation preserving homeomorphism near ζ. Thus we have $\{\cup_{n=0}^{\infty} f^{-n}(\zeta)\} \cap \{\cup_{i=1}^{p} \partial B_i\} = \emptyset$. Let E be the complementary component of $\overline{B_i}$ which contains ζ. From the above, we have

$$\left(\cup_{i=1}^{3} \cup_{n=0}^{1} f^{-n}(D_i)\right) \cup \left(\cup_{n=0}^{1} f^{-n}(\zeta)\right) \subset E.$$

Thus, by induction, we easily obtain

$$\left(\cup_{i=1}^{3} \cup_{n=0}^{\infty} f^{-n}(D_i)\right) \cup \left(\cup_{n=0}^{\infty} f^{-n}(\zeta)\right) \subset E.$$

Setting $g = f^p$, we have $g(B_i) = B_i$ and $g^{-1}(D_1) = \cup_{n=0}^{p} f^{-n}(D_1)$. Since $g^{-1}(D_1) \subset E$, by Lemma 6, the boundary of B_j is a Jordan curve. $\square$

The theorem above implies the boundary of each component of the Fatou set of Newton's method for $z^3 - 1 = 0$ is a Jordan domain. The argument similar to that in the proof shows the following.

Theorem 8 *Let a be a non-zero complex number and $g(z) = z^n - a$. Set $f(z) = z - (g(z)/g'(z))$. Then $F(f)$ consists of Jordan domains.*

Proof. If $n = 2$, then it is clear. Because $F(f)$ consists of two components.

We consider the case $n \geq 3$. In fact, we have

$$f(z) = \frac{(n-1)z^n + a}{nz^{n-1}}.$$

From this, we see that $\sqrt[n]{a}$ is a critical point and a super-attracting fixed point, that 0 is a critical point with multiplicity $n-2$, that ∞ is a only one repelling fixed point and that $f(0) = \infty$. Shishikura ([12]) showed that the Julia set of Newton's method for a non-constant polynomial is connected. We denote by F_ζ the Fatou component containing ζ. The cyclic components of $F(f)$ are $F_{\sqrt[n]{a}}$ only and $\partial F_{\sqrt[n]{a}}$ and the boundary of each component of $f^{-1}(F_{\sqrt[n]{a}})$ contains ∞. By the argument similar to that in the proof of Theorem 7, we obtain $\partial F_{\sqrt[n]{a}}$ is a Jordan curve (see also Figure 1). $\square$

5 A Sierpinski carpet

We say that a closed subset in $\hat{\mathbf{C}}$ is a Sierpinski carpet if it is the complement of a countable dense family of open topological discs whose diameters tend to zero and whose closures are pairwise disjoint closed topological discs. In [5], Milnor and Tan Lei gave a quadratic rational function whose Julia set is a Sierpinski carpet and Pilgrim ([10]) also gave a rational function of degree three whose Julia set is a Sierpinski carpet.

Example 9 *The Julia set of*

$$f(z) = 27\frac{z^2(z-1)}{(3z-2)^2(3z+1)}$$

is a Sierpinski carpet.

Indeed, the critical points of f is 0, $2/3$, ∞ and ∞ and we have

$$f(\frac{2}{3}) = \infty, \qquad f(\infty) = 1, \qquad f(1) = 0 \quad \text{and} \quad f(0) = 0.$$

Hence f is a postcritically finite hyperbolic rational function. It was stated in [7] that the Julia set of a postcritically finite rational function is connected. Thus $J(f)$ is connected. The origin is the only one attracting fixed point and every component of $F(f)$ is eventually mapped onto F_0. Let E be the component of $(\overline{F_0})^c$ which contains F_1, which is the only one component of $f^{-1}(F_0)$ not coinciding with F_0. Then we see $f(\overline{E}) = \hat{\mathbf{C}}$. It follows that E contains a component of $f^{-1}(F_1)$. Since ∞ is a critical point with multiplicity two, F_∞ is the only one component of $f^{-1}(F_1)$. Hence, from Lemma 6, ∂F_0 is a Jordan curve and thus the boundary of each component of $F(f)$ is a Jordan curve.

Suppose that there exist components D_1 and D_2 of $F(f)$ such that $D_1 \neq D_2$ and $\partial D_1 \cap \partial D_2 \neq \emptyset$. Then we may also assume that $f(D_1) = f(D_2)$ by replacing D_1 and D_2 by some $f^n(D_1)$ and $f^n(D_2)$. Let z be a point in $\partial D_1 \cap \partial D_2$, then z is either a critical point of f or not a simple point with respect to $f(D_1)$. This is contradiction. Thus we see that $J(f)$ is a Sierpinski carpet.

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