Local connectedness of Julia sets for transcendental entire functions

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1 Introduction

Let f be a transcendental entire function. We denote the n-th iteration of f by f^n. The maximal open subset of the complex plain $\mathbb{C}$ where $\{f^n\}$ is normal is called the Fatou set of f and is denoted by $F(f)$. The complement of $F(f)$ in $\widehat{\mathbb{C}}$ is called the Julia set of f and is denoted by $J(f)$. Note that we regard ∞ is a point in the Julia set. Since $J(f)$ is completely invariant, $J(f) \setminus \{\infty\}$ is unbounded in $\mathbb{C}$. Kisaka considered a topological property of the boundary of unbounded Fatou components ([6]). In this paper, we consider bounded Fatou components. Stallard ([10]) showed that if a transcendental entire function has small growth order, then every Fatou component is bounded.

We call $\zeta \in \mathbb{C}$ a singular value if f is not a smooth covering map over any neighborhood of ζ. It plays an important role in studying its dynamics. We denote the set of them by $\mathrm{sing}(f^{-1})$ and set

$$\mathcal{S} = \{f \mid f \text{ is transcendental entire and } \mathrm{sing}(f^{-1}) \text{ is a finite set}\}.$$

For $f \in \mathcal{S}$, every point of $\mathrm{sing}(f^{-1})$ is either an asymptotic value of f or a critical value of f. It is known that if $f \in \mathcal{S}$, then $F(f)$ does not contain neither a wandering domain nor a Baker domain ([4]). Hence if $f \in \mathcal{S}$, then we have $\{z \mid f^n(z) \to \infty\} \subset J(f)$.

In § 2, we give two conditions on singular values to treat Julia sets easier just like those of hyperbolic rational functions. Under those conditions, we show that the boundary of a bounded Fatou component is a Jordan curve and further that if all the cyclic components of the Fatou set are bounded, then its Julia set is locally connected. In § 3, we consider the family $\{f_\lambda(z) = \lambda z e^z \mid \lambda \in \mathbb{C}\}$, which is contained in $\mathcal{S}$. By using curves in Julia sets, we show that some functions in the family have only bounded Fatou components. Curves in Julia sets of exponential maps were first studied by Devaney and Krych ([2]) as Cantor bouquets. Finally, in § 4, we give an example of a transcendental entire function whose Julia set is a Sierpinski carpet.

Some pictures of Fatou set stated in this note can be seen in the author's home page[1].

2 Local connectedness

For a transcendental entire function f, we give two conditions on singular values.

[SV1] If $\zeta \in F(f) \cap \mathrm{sing}(f^{-1})$, then it is a critical value and is absorbed by an attracting cycle.
[SV2] If $\zeta \in J(f) \cap \mathrm{sing}(f^{-1})$, then

$$\overline{\cup_{n\geq 0} f^n(\zeta)} \cap \partial D = \emptyset$$

for any Fatou component D.

Set

$$\mathcal{SV} = \{f \in \mathcal{S} \mid f \text{ satisfies [SV1] and [SV2] }\}.$$

Note that, for $f \in \mathcal{SV}$, every cyclic component of $F(f)$ is attracting. We also remark that, for $f \in \mathcal{SV}$, it is not necessary that its cyclic components are bounded. For example, let $f(z) = \lambda \sin z$ with $\lambda \in \mathbb{R}$, $|\lambda| < 1$. Then $f \in \mathcal{SV}$ and $F(f)$ has an unbounded attracting component.

Theorem 1 *Let D be a cyclic Fatou component of $f \in \mathcal{SV}$. If D is bounded, then ∂D is a Jordan curve.*

Proof. It is enough to show the case that D is a fixed component. By using the argument similar to that used to prove that the boundary of a simply connected attracting component of a hyperbolic rational function is a closed curve (see e.g. [1]), we prove the theorem.

First, we show

$$\lim_{n\to\infty} \inf_{z\in\partial D} |(f^n)'(z)| = +\infty.$$

Assume that there exist $\epsilon > 0$ and an infinite set N of positive integers such that, for $n \in N$, $z_n \in \partial D$ and branches g_n of $(f^n)^{-1}$ at z_n with $g_n(z_n) \in \partial D$

[1] http://www.math.kochi-u.ac.jp/morosawa/indexe.html

satisfying $|g_n'(z_n)| > \epsilon$. If necessary, taking subsequence, we may assume $z_n \to \zeta \in \partial D$. Take a neighborhood U of ζ so that

$$\overline{\cup_{n\geq 0} f^n(\mathrm{sing}(f^{-1}))} \cap U = \emptyset.$$

Choose a branch h_n of $(f^n)^{-1}$ in U so that $h_n(z_n) = g_n(z_n)$ for $z_n \in U$. Since U intersects the Julia set, an arbitrary locally uniform limit of a subsequence of $\{h_n\}$ is constant. Further, it is not infinity, for D is bounded. Hence we have $h_n'(z) \to 0$ locally uniformly in U, which implies $g_n'(z_n) \to 0$. This contradicts our assumption.

Take $w \in \partial D$ and set $z_n = f^n(w)$. Choose a branch g_n of $(f^n)^{-1}$ with $g_n(z_n) = w$. We have

$$f^n(g_n(z)) = z \quad \text{and} \quad |(f^n)'(g_n(z))||g_n'(z)| = 1$$

for all z in some neighborhood of z_n. From the above, for all $\epsilon > 0$, there exists n_0 such that

$$1 = |(f^n)'(w)||g_n'(z_n)| < \epsilon |(f^n)'(w)|$$

for all $n \geq n_0$. Hence we have $|(f^n)'(w)| > 1/\epsilon$.

Since D is an attracting component of a transcendental entire function, it is connected. The assumption that $f \in \mathcal{SV}$ implies that f is a finite branched covering of D. We denote the local degree of f in D by p. Hence $f|_D$ is conformally conjugate to a finite Blaschke product of degree p in the unit disk. Thus there exist a compact set E of D, $A = \{z \mid r < |z| < 1\}$ and a homeomorphism φ from A onto $D \setminus E$ which satisfies

$$f \circ \varphi(z) = \varphi(z^p)$$

for $z \in \{z \mid r^{1/p} < |z| < 1\}$. Since f is expanding on some neighborhood of ∂D as we show above, φ can be extended continuously to a map from $A' = \{z \mid r < |z| \leq 1\}$ onto $\overline{D} \setminus E$. We again denote it by φ. Hence $\partial D = \varphi(S^1)$ is a closed curve, where $S^1 = \{z \mid |z| = 1\}$.

Assume that there exist distinct points z_1 and z_2 in S^1 such that $\varphi(z_1) = \varphi(z_2)$. Take a simple curve γ in A' whose end points are z_1 and z_2. Then $\Gamma = \varphi(\gamma)$ is a simple closed curve in $\overline{D}$. Since $\{f^n\}$ is bounded on Γ, from the maximum modulus theorem, it is also bounded on the bounded complementary component G of Γ. This contradicts that G intersects the Julia set. Hence ∂D is a Jordan curve. ■

Theorem 2 *Let f be in $\mathcal{SV}$. If all the cyclic components of $F(f)$ are bounded, then $J(f)$ is locally connected.*

Proof. A theorem due to Whyburn ([13]) shows that a closed set in $\widehat{\mathbb{C}}$ is locally connected if and only if it satisfies following two conditions.
(1) The boundary of each complementary component of it is locally connected.
(2) For an arbitrary $\epsilon > 0$, the number of the complementary components whose diameters with respect to the spherical distance exceed ϵ is finite.

From theorem 1, the boundary of each cyclic component is locally connected. Since f is in $\mathcal{SV}$, there exist no wandering components. Hence, an arbitrary component of $F(f)$ is mapped to some cyclic component by finitely many iteration of f. Thus its boundary is locally connected because of the condition [SV2].

Next, we show $J(f)$ satisfies the condition (2). Take a cyclic component D. Since $\overline{D}$ does not contain any asymptotic value, an arbitrary component of $f^{-n}(D)$ is bounded for all n. We show that, for an arbitrary $\epsilon > 0$, the number of the components of the backward orbit of D whose diameters exceed ϵ is finite, because the number of the cyclic component of $F(f)$ is finite for $f \in \mathcal{S}$. Assume that there exist a component D_0 of $F(f)$ and components D_n $(n \geq 1)$ of $F(f)$ each of whose diameter exceeds ϵ such that

$$f^{j(n)}(D_n) = D_0 \qquad \text{and} \qquad (\cup_{n\geq 1} D_n) \cap (\cup f^n(\text{sing}(f^{-1})) = \emptyset.$$

From the fact that ∂D_0 is a Jordan curve and the assumption that f satisfies [SV2], we can take a neighborhood U of $\overline{D_0}$ such that a branch g_n on U of $f^{-j(n)}$ satisfying $g_n(D_0) = D_n$. Then $\{g_n\}$ is a normal family on U. Since U intersects the Julia set, every limit function of it is constant. This is a contradiction. ■

3 Julia sets of $f_\lambda(z) = \lambda z e^z$

We set

$$\mathcal{D} = \{f_\lambda(z) = \lambda z e^z \mid \lambda \in \mathbb{C}\}.$$

Iteration of elements of $\mathcal{D}$ are studied by some mathematicians (e.g. [5], [7] and [11]). In the case that $\lambda = 0$, we regard that f_0 has an attracting fixed point 0 for convenience. For $f_\lambda(z) \in \mathcal{D}$, we see

$$\text{sing}(f_\lambda^{-1}) = \left\{ f_\lambda(-1) = \frac{\lambda}{e},\ 0 \right\}$$

and the set of the fixed points of f_λ is

$$\{0\} \cup \{-\text{Log}\ \lambda + 2n\pi i\}_{n\in\mathbb{Z}}.$$

We define

$$B = \{\lambda \in \mathbb{C} \mid f_\lambda^n(-1) \not\to \infty\}$$

and further

$$B_{-1} = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \quad \text{and} \quad B_1 = \{\lambda \in \mathbb{C} \mid |1 - \mathrm{Log}\ \lambda| < 1\}.$$

Proposition 3 *If $\lambda \notin B$, then $J(f_\lambda) = \widehat{\mathbb{C}}$. The function $f_\lambda(z)$ has an attracting fixed point if and only if $\lambda \in B_{-1} \cup B_1$. Further, if $\lambda \in B_{-1}$, then 0 is the attracting fixed point and $f_\lambda^n(-1)$ tends to 0 as n tends to infinity and if $\lambda \in B_1$, then $-\mathrm{Log}\ \lambda$ is the attracting fixed point.*

Proof. Since fixed points of f_λ are 0 and $-\mathrm{Log}\ \lambda + 2n\pi i$, it is clear that f_λ has an attracting fixed point if and only if $\lambda \in B_{-1} \cup B_1$ and if $\lambda \in B_{-1}$, then 0 is the attracting fixed point and if $\lambda \in B_1$, then $-\mathrm{Log}\ \lambda$ is the attracting fixed point.

We denote a neighborhood of the origin by $U_\epsilon = \{z \mid |z| < \epsilon\}$. In the case that $\lambda \in B_{-1}$, we can choose ϵ so small that $f^{-1}(U_\epsilon)$ consists of two components and one of which contains U_ϵ and the other does not intersect U_ϵ, because the origin is an asymptotic value and has only one inverse image. This means the attracting component containing the origin is completely invariant, since f_λ is not univalent on it. Hence it contains the critical point of f, which is -1.

If $\lambda \notin \overline{B_{-1}}$, then the asymptotic value 0 is a repelling fixed point. Since f_λ has only two singular values $f_\lambda(-1)$ and 0, we have $J(f_\lambda) = \widehat{\mathbb{C}}$ for $\lambda \notin B$ by the relations between cyclic Fatou components and singular values. ■

Let

$$\Lambda = \{\lambda \in \mathbb{C} \mid |\mathrm{Im}\ \lambda| \geq e\mathrm{Arg}\ \lambda\}.$$

In [5], Fagella showed the following.

Theorem 4 *If $\lambda \in \Lambda$, then there exists a curve $L(t)$ ($t \in [0, \infty)$) satisfying the following conditions.*
(1) $f_\lambda(L(t)) = L(t)$.
(2) *For sufficiently large h*

$$L(t) \cap H_h \subset \{z \mid -\pi - \theta \leq \mathrm{Im}\ z \leq \pi - \theta\},$$

where $H_h = \{z \mid \mathrm{Re}\ z \geq h\}$ and $\theta = \mathrm{Arg}\ \lambda$.
(3) $L(0) = 0$.
(4) *For $t \neq 0$*

$$\lim_{n\to\infty} f_\lambda^n(L(t)) = \infty.$$

Theorem 5 *If $\lambda \in \Lambda$ and $f_\lambda(z) = \lambda z e^z$ has an attracting cycle whose period is greater than one, then each component of $F(f_\lambda)$ is bounded and $J(f_\lambda)$ is locally connected.*

Proof. Assume that the period of the attracting cycle is $p(>1)$. Let D_0 be the attracting component of this cycle containing -1. If ∂D_0 contains the fixed point 0, then so does each $\partial f_\lambda^k(D_0)$ for $k = 1, 2, \cdots, p-1$. Hence each $f_\lambda^k(D_0)$ contains a curve terminating at 0 satisfying that the curves are permuted by f_λ. From Theorem 4, there exists an invariant curve whose end point is 0, which we denote by L. This is a contradiction. Hence there exists a neighborhood $U_\epsilon = \{z \mid |z| < \epsilon\}$ of 0 such that U_ϵ does not intersect each of the attracting cyclic components. If necessary, we take ϵ sufficiently small so that $f_\lambda^{-1}(U_\epsilon)$ consists of two simply connected components. We denote the unbounded one by H, which is in the left half plane. Each component of the inverse image of L tends to infinity in both direction and intersects with H in one direction. Further, for all sufficiently large h and $n \in \mathbb{Z} \setminus \{0\}$

$$\{z \mid \mathrm{Re}\ z > h \quad \text{and} \quad 2n\pi - \theta < \mathrm{Im}\ z < 2(n+1)\pi - \theta\}$$

intersects a component of the inverse image of L, where recall $\theta = \mathrm{Arg}\ \lambda$. Choose l_1 and l_2 from the components of the inverse image of L such that the domain bounded by them contains -1 and does not contain any component of it. Since l_1 and l_2 is in $J(f_\lambda)$, D_0 is contained in the domain bounded by l_1, l_2 and ∂H. Assuming that D_0 is unbounded and hence so is $D_k = f_\lambda^k(D_0)$ for $k = 1, 2, \cdots, p-1$, we have

$$\left(\cup_{k=0}^{p-1} D_k\right) \cap \{z \mid \mathrm{Re}\ z > h\} \subset \{z \mid \mathrm{Re}\ z > h \quad \text{and} \quad |\mathrm{Im}\ z| < M\}$$

for some M. Choose h_1 and set

$$R(h_1) = \{z \mid \mathrm{Re}\ z > h_1 \quad \text{and} \quad |\mathrm{Im}\ z| < M\}$$

so that

$$|f_\lambda(z)| = |\lambda||z|e^{\mathrm{Re}\ z} > \mathrm{Re}\ z + 2M$$

for every $z \in R(h_1)$. Hence we have $\mathrm{Re}\ f_\lambda^p(z) > \mathrm{Re}\ z$ for every $z \in D_0 \cap R(h_1)$. This implies $\lim_{n\to\infty} f^{np}(z) = \infty$ for $z \in D_0 \cap R(h_1)$. On the other hand, for $z \in D_0$, $f^{np}(z)$ tends to the attracting fixed point of f^p in D_0. This is a contradiction. Thus D_0 is bounded and so is D_k. Since $\mathrm{sing}(f_\lambda^{-1}) = \{0, f_\lambda(-1)\}$ and the asymptotic value is a repelling fixed point, $\{D_k\}_{k=0}^{p-1}$ is the only one set of cyclic components $F(f_\lambda)$ contains. By Theorem 2, we see that $J(f_\lambda)$ is locally connected. ∎

We call the subset of the Julia set whose points do not lie on the boundary of any component of the Fatou set the residual Julia set. Residual Julia sets for rational functions are considered in [9] and those for entire functions are considered in [3].

Theorem 6 *If $\lambda \in \Lambda$ and $f_\lambda(z) = \lambda z e^z$ has an attracting cycle whose period is greater than one, then $L(t)$ defined in Theorem 4 is contained in the residual Julia set of $J(f_\lambda)$.*

Proof. As was shown in the proof of Theorem 5, a fixed point $L(0) = 0$ is not on any cyclic component of $F(f_\lambda)$. Since any component of $F(f_\lambda)$ is eventually mapped into the immediate basin containing -1, 0 is not on the boundary of any Fatou component.

Theorem 5 shows that every component of $F(f_\lambda)$ is bounded. Further it is not wandering. On the other hand, for $t \neq 0$, $f_\lambda^n(L(t))$ tends to infinity. Hence $L(t)$ does not intersect the boundary of any component of $F(f_\lambda)$. ■

4 Another example

We say that a closed subset in $\widehat{\mathbb{C}}$ is a Sierpinski carpet if it is the complement of a countable dense family of open topological discs whose diameters tend to zero and whose closure are pairwise disjoint closed topological discs. Note that any two Sierpinski carpets are homeomorphic. Some rational functions whose Julia sets are Sierpinski carpets are known (see [8] [12]). We give an example of a transcendental entire function whose Julia set is a Sierpinski carpet by the argument similar to that in the previous section.

Theorem 7 *Let*

$$g_a(z) = ae^a\{z - (1-a)\}e^z$$

for $a > 1$. Then $J(g_a)$ is locally connected. Moreover, it is a Sierpinski carpet.

Proof. It is clear that $\text{sing}(g_a^{-1}) = \{-a,\ 0\}$, where $-a$ is a critical value and 0 is an asymptotic value. Hence $g_a \in \mathcal{S}$. Since $g_a(-a) = -a$, $-a$ is a superattracting fixed point. We denote the attracting component containing $-a$ by D. By using the graphical analysis, it is easy to see that $\lim_{n\to\infty} g_a^n(x) = \infty$ for all $x > b$, where b is the fixed point satisfying $-a < b < 0$. Hence the interval $[b,\ \infty)$ in $\mathbb{R}$ is contained in $J(g_a)$ and especially the asymptotic value 0 is contained in $J(g_a)$.

Set

$$\begin{aligned} l^+ &= \left\{ z = x + iy \mid x = 1 - a - y\frac{\cos y}{\sin y},\ \pi < y < 2\pi \right\} \quad \text{and} \\ l^- &= \left\{ z = x + iy \mid x = 1 - a - y\frac{\cos y}{\sin y},\ -2\pi < y < -\pi \right\}. \end{aligned}$$

Then each of $g_a(l^+)$ and $g_a(l^-)$ is a positive real axis and hence is contained in $J(g_a)$. It is easy to see

$$\begin{aligned} \lim_{y\to\pi+0} 1 - a - y\frac{\cos y}{\sin y} &= -\infty & \lim_{y\to 2\pi-0} 1 - a - y\frac{\cos y}{\sin y} &= +\infty \\ \lim_{y\to-\pi-0} 1 - a - y\frac{\cos y}{\sin y} &= -\infty & \lim_{y\to -2\pi+0} 1 - a - y\frac{\cos y}{\sin y} &= +\infty. \end{aligned}$$

Denoting the domain bounded by l^+ and l^- by E, we have $E \supset D$. Choose h so that $|g_a(z)| \geq |z| + 4\pi$ for all z satisfying Re $z > h$. Because $g_a^n(z)$ tend to $-a$ for every $z \in D$, we have Re $z < h$ for $z \in D$. Let N be the smallest number such that $g_a^N(0) > h$. Choose $\epsilon > 0$ so small that $g_a^N(U_\epsilon) \subset H_h$, where $U_\epsilon = \{z \mid |z| < \epsilon\}$ and $H_h = \{z \mid \text{Re } z > h\}$. This implies $D \cap U_\epsilon = \emptyset$. Let K be an unbounded component of $g_a^{-1}(U_\epsilon)$. Then ∂K intersects with l^+ and l^-. Hence we see that D is bounded.

Since D is the only one cyclic component of $F(g_a)$ and $F(g_a)$ has no wandering components, every components of $F(g_a)$ is eventually mapped to D. From Theorem 2, we see that $J(g_a)$ is locally connected, for D is an attracting component.

Assume that there exist two components D_1 and D_2 of $F(g_a)$ such that $\partial D_1 \cap \partial D_2 \neq \emptyset$ and take $\zeta \in \partial D_1 \cap \partial D_2$. Since they are eventually mapped to D, we can take M such that

$$g_a^M(D_1) \cap g_a^M(D_2) = \emptyset \quad \text{and} \quad g_a^{M+1}(D_1) \cap g_a^{M+1}(D_2) \neq \emptyset.$$

This implies $g_a^M(\zeta)$ is a critical point of g_a. This is a contradiction. Therefore $J(g_a)$ is a Sierpinski carpet. ∎

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