Bounded Fatou components of transcendental entire functions

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1 Introduction

Let f be a transcendental entire function. We denote the n-th iteration of f by f^n. The maximal open subset of the complex plain $\mathbb{C}$ where $\{f^n\}$ is normal is called the Fatou set of f and is denoted by $F(f)$. The complement of $F(f)$ in $\widehat{\mathbb{C}}$ is called the Julia set of f and is denoted by $J(f)$. Note that we regard ∞ is a point in the Julia set. Since $J(f)$ is completely invariant, $J(f) \setminus \{\infty\}$ is unbounded in $\mathbb{C}$. Kisaka considered a topological property of the boundary of unbounded Fatou components ([7]). In this paper, we consider bounded Fatou components. Stallard ([12]) showed that if a transcendental entire function has small growth order, then every Fatou component is bounded. Another way to find bounded Fatou components is to find a polynomial-like mapping. If its filled-in Julia set has interior points, then the Fatou set contains a bounded component.

We call $\zeta \in \mathbb{C}$ a singular value if f is not a smooth covering map over any neighborhood of ζ. It plays an important role in studying its dynamics. We denote the set of them by $\mathrm{sing}(f^{-1})$ and set

$$\mathcal{S} = \{f \mid f \text{ is transcendental entire and } \mathrm{sing}(f^{-1}) \text{ is a finite set.}\}.$$

For $f \in \mathcal{S}$, every point of $\mathrm{sing}(f^{-1})$ is either an asymptotic value of f or a critical value of f. It is known that if $f \in \mathcal{S}$, then $F(f)$ does not contain neither a wandering domain nor a Baker domain ([4]). Hence if $f \in \mathcal{S}$, then we have

$$\{z \mid f^n(z) \to \infty\} \subset J(f).$$

In § 2, we give an example of a transcendental entire function whose Fatou set contains bounded and unbounded components.

In § 3, we consider the family $\{f_\lambda(z) = \lambda z e^z \mid \lambda \in \mathbb{C}\}$, which is contained in $\mathcal{S}$. We see that some functions in the family have bounded Fatou components.

Finally, in § 4, we give an example of a transcendental entire function whose Julia set is a Sierpinski carpet.

Proofs of theorems in this note are given in [11].

Some pictures of Fatou set stated in this note can be seen in the author's home page[1].

2 An example obtained by a polynomial-like mapping

We give an example of a transcendental entire function whose Fatou set contains a bounded component by using a polynomial-like mapping.

Example 1 *Let*

$$f(z) = z^2 \exp \frac{1}{2}(1 - z^4).$$

Then the attracting component containing a super-attracting fixed point 0 *is bounded.*

Indeed, set

$$U = \left\{ z \,\middle|\, |z| < \frac{1}{2} \right\}$$

and $V = f(U)$. Elementary calculation shows that $f(\partial U) = \partial V$ and $U \subset \overline{U} \subset V$. Hence $(f,\ U,\ V)$ is a polynomial-like mapping of degree two and the filled-in Julia set of $(f,\ U,\ V)$ contains the attracting component containing 0, which we denote by D_0. Note that 0 is both a critical point and an asymptotic value of f. Hence, the components of $f^{-1}(D_0)$ except D_0 are unbounded.

[1] http://www.math.kochi-u.ac.jp/morosawa/indexe.html

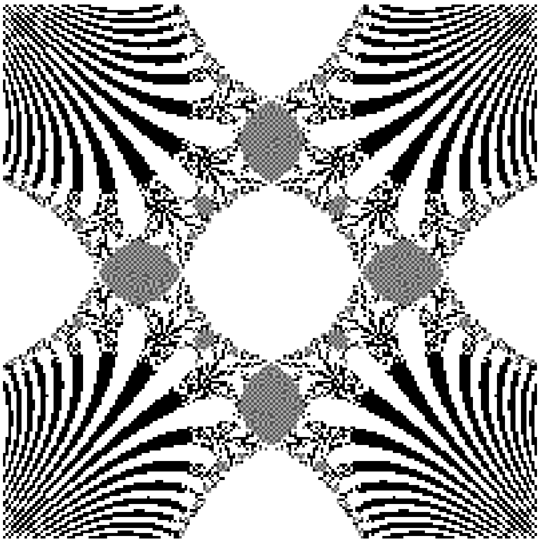

Figure 1: $f(z) = z^2 \exp(1 - z^4)/2$

3 Julia sets of $f_\lambda(z) = \lambda z e^z$

We set

$$\mathcal{D} = \{f_\lambda(z) = \lambda z e^z \mid \lambda \in \mathbb{C}\}.$$

Iteration of elements of $\mathcal{D}$ are studied by some mathematicians (e.g. [5], [6], [8] and [13]). In the case that $\lambda = 0$, we regard that f_0 has an attracting fixed point 0 for convenience. For $f_\lambda(z) \in \mathcal{D}$, we see

$$\mathrm{sing}(f_\lambda^{-1}) = \left\{ f_\lambda(-1) = \frac{\lambda}{e},\ 0 \right\}$$

and the set of the fixed points of f_λ is

$$\{0\} \cup \{-\mathrm{Log}\ \lambda + 2n\pi i\}_{n\in\mathbb{Z}}.$$

We define

$$B = \{\lambda \in \mathbb{C} \mid f_\lambda^n(-1) \not\to \infty\}$$

and further

$$\begin{aligned} B_{-1} &= \{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \quad \text{and} \\ B_1 &= \{\lambda \in \mathbb{C} \mid |1 - \log \lambda| < 1\}. \end{aligned}$$

Since fixed points of f_λ are 0 and $-\text{Log}\ \lambda + 2n\pi i$, it is clear that f_λ has an attracting fixed point if and only if $\lambda \in B_{-1} \cup B_1$ and if $\lambda \in B_{-1}$, then 0 is the attracting fixed point and if $\lambda \in B_1$, then $-\text{Log}\ \lambda$ is the attracting fixed point.

In the case that $\lambda \in B_{-1}$, the attracting component containing the origin is completely invariant, since f_λ is not univalent on it. Hence it contains the critical point of f, which is -1.

If $\lambda \notin \overline{B_{-1}}$, then the asymptotic value 0 is a repelling fixed point. Since f_λ has only two singular values $f_\lambda(-1)$ and 0, we have $J(f_\lambda) = \widehat{\mathbb{C}}$ for $\lambda \notin B$ by the relations between cyclic Fatou components and singular values. Hence we have the following proposition.

Proposition 1 *If $\lambda \notin B$, then $J(f_\lambda) = \widehat{\mathbb{C}}$. The function $f_\lambda(z)$ has an attracting fixed point if and only if $\lambda \in B_{-1} \cup B_1$. Further, if $\lambda \in B_{-1}$, then 0 is the attracting fixed point and $f_\lambda^n(-1)$ tends to 0 as n tends to infinity and if $\lambda \in B_1$, then $-\text{Log}\ \lambda$ is the attracting fixed point. Further, if f_λ has an attracting cycle, then $F(f_\lambda)$ is equal to the basin of the attracting cycle.*

We define the following set

$$\Lambda = \{\lambda \in \mathbb{C} \mid |\text{Im}\ \lambda| \geq e\text{Arg}\ \lambda\}.$$

Theorem 2 *If $\lambda \in \Lambda$ and $f_\lambda(z) = \lambda z e^z$ has an attracting cycle whose period is greater than one, then each component of $F(f_\lambda)$ is bounded.*

To prove the theorem above, we need the following theorem proved by Fagella ([5]).

Theorem 3 *If $\lambda \in \Lambda$, then there exists a curve $L(t)$ $(t \in [0, \infty))$ satisfying the following conditions.*
(1) $f_\lambda(L(t)) = L(t)$.
(2) *For sufficiently large h*

$$L(t) \cap H_h \subset \{z \mid -\pi - \theta \leq \text{Im}\ z \leq \pi - \theta\},$$

where $H_h = \{z \mid \text{Re}\ z \geq h\}$ and $\theta = \text{Arg}\ \lambda$.
(3) $L(0) = 0$.
(4) *For $t \neq 0$*

$$\lim_{n\to\infty} f_\lambda^n(L(t)) = \infty.$$

Curves in Julia sets of exponential maps were first studied by Devaney and Krych ([2]) as Cantor bouquets. In [5], Fagella called this $L(t)$ a fixed hair.

Outline of proof of Theorem 2. Assume that the period of the attracting cycle is $p(>1)$. Let D_0 be the attracting component of this cycle containing -1. From Theorem 3, there exists an invariant curve whose end point is 0, which we denote by L. Hence ∂D_0 does not contain the fixed point 0, because the period of attracting cycle is greater than 1. We choose a neighborhood $U_\epsilon = \{z \mid |z| < \epsilon\}$ so that U_ϵ does not intersect D_0 and that $f_\lambda^{-1}(U_\epsilon)$ consists of two simply connected components. We denote the unbounded component of $f_\lambda^{-1}(U_\epsilon)$ by H. It is clear that $H \cap D_0 = \emptyset$ and that H contains some left half plane $\{z \mid \text{Re } z < a\}$. Choose l_1 and l_2 from the components of the inverse image of L such that the domain bounded by them contains -1 and does not contain any component of it. Note that each l_i $(i = 1, 2)$ intersects both H and H_h in Theorem 3. Since l_1 and l_2 are in $J(f_\lambda)$, D_0 is contained in the domain bounded by l_1, l_2 and ∂H. Assume that D_0 is unbounded. Then it is in some strip in the right half plane by Theorem 3. Hence we see that $\{f^{np}(z)\}$ tends to infinity for every z in D_0 whose real part is sufficiently large. This is a contradiction. Hence D_0 is bounded. Since asymptotic value is not on the boundary of any Fatou component, every component of the basin of the attracting cycle is bounded. ∎

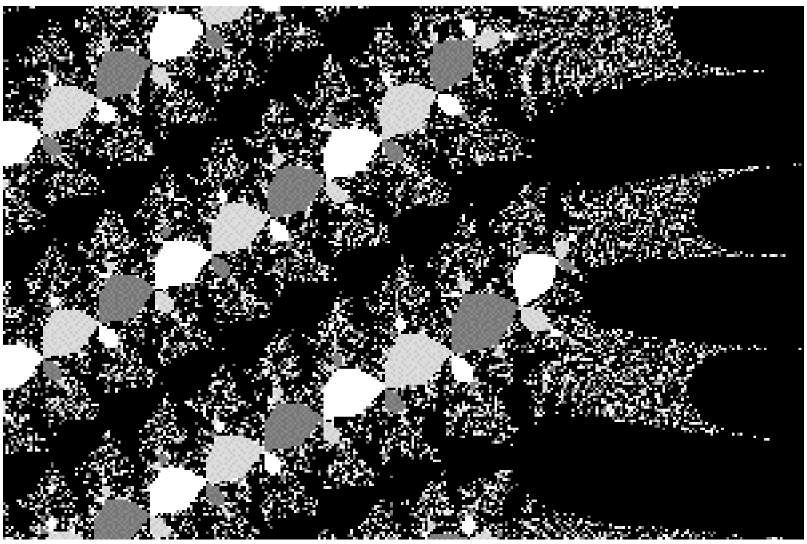

Figure 2: $f_\lambda(z) = \lambda z e^z$; $\lambda = 2.68699 + 3.87109i$

Remark 1 Further, we see that $J(f_\lambda)$ is locally connected if f_λ satisfies the

condition in Theorem 2.
Remark 2 The curve L in the proof does not intersect the boundary of any Fatou component. We call the subset of the Julia set whose points do not lie on the boundary of any component of the Fatou set the residual Julia set. Residual Julia sets for rational functions are considered in [10] and those for entire functions are considered in [3].

4 Another example

We say that a closed subset in $\widehat{\mathbb{C}}$ is a Sierpinski carpet if it is the complement of a countable dense family of open topological discs whose diameters tend to zero and whose closure are pairwise disjoint closed topological discs. Note that any two Sierpinski carpets are homeomorphic. Some rational functions whose Julia sets are Sierpinski carpets are known (see [9] [14]). We give an example of a transcendental entire function whose Julia set is a Sierpinski carpet by the argument similar to that in the previous section.

Theorem 4 *Let*

$$g_a(z) = ae^a\{z-(1-a)\}e^z$$

for $a>1$. *Then* $J(g_a)$ *is a Sierpinski carpet.*

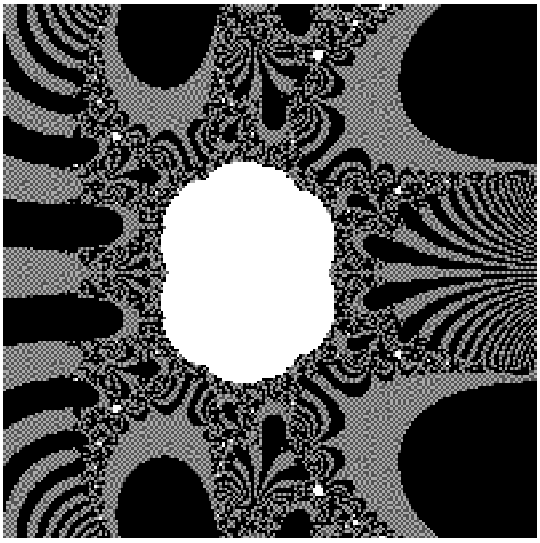

Figure 3: $g_a(z) = ae^a\{z-(1-a)\}e^z$; $a=2$

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