An example of cyclic Baker domains

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Abstract

In this note, we give an concrete example of cyclic Baker domains.

1 Introduction

Let f be a transcendental entire function. We denote the n-th iteration of f by f^n. The maximal open subset where $\{f^n\}$ is normal is called the Fatou set of f and is denoted by $F(f)$. The complement of $F(f)$ is called the Julia set of f and is denoted by $J(f)$. We say that a component D of $F(f)$ is forward invariant if $f(D) \subset D$. In the case of rational functions, a forward invariant component is one of the following four: an attracting component, a parabolic component, a Siegel disc and a Herman ring. We say that a component D of $F(f)$ is wandering if $f^n(D) \neq f^m(D)$ for every n and m with $n \neq m$. The Fatou set of any rational function does not contain wandering domains (see [1], [4]). If a point over any whose neighborhood a holomorphic map is not a smooth covering map, then it is called a singular value of the map. For rational functions, it is a critical value of it. Singular values play important roles in studying its dynamics. If the set of all the singular values of a transcendental entire function is finite, then its dynamics is as same as that of rational functions as the following sense. Any invariant component of its Fatou set is one of the four above and there is no wandering domain (see [2]). On the contrary, if the set of the singular values is infinite, then other situations may occur. We say that a forward invariant component D of the Fatou set of a transcendental entire function f is a Baker domain, if $\{f^n\}$ converges to infinity uniformly on compact sets in D. In this note, we give a concrete example of cyclic Baker domains.

See, for example, [3] to find the properties of iteration of transcendental entire functions which we use in this note.

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2 Cyclic Baker domains

Theorem 1 *Let* $f(z) = z\exp(2\pi i/p)(2+\exp(-z^p))$ *for* $p \geq 2$*, then* $F(f)$ *has Baker domains with period* p*.*

Proof. First, we show that $g(z) = z(2+\exp(-z^p))$ has at least p Baker domains.
Take θ with $0 < \theta < \pi/2p$ and $R_1 > 3/(2\cos p\theta)$ and set

$$A = \{z \mid |\arg z| < \theta \quad \text{and} \quad |z| > R_1\}.$$

For every $z \in A$, we have

$$\mathrm{Re}\ z^p = |z|^p \cos p\theta > \frac{3}{2}|z| \quad \text{and} \quad |\exp(-z^p)| < \frac{1}{2}.$$

Hence, for $z \in A$, we have

$$|\arg(z + \exp(-z^p))| < c|\exp(-z^p)|$$

for some c. For this c, we take $R_2 > R_1$ such that

$$c\sum_{n=0}^{\infty} \exp\left(-\left(\frac{3}{2}\right)^n R_2\right) < \theta.$$

Choose φ satisfying

$$0 < \varphi < \theta - c\sum_{n=0}^{\infty} \exp\left(-\left(\frac{3}{2}\right)^n R_2\right)$$

and set

$$B_0 = \{z \mid |\arg z| < \varphi \quad \text{and} \quad |z| > R_2\}.$$

We show that

$$\{g^n(z)\}_{n=0}^{\infty} \subset A$$

for every $z \in B_0$. Assuming that $g^k(z) \in A$ for $k = 0,\ 1, \cdots,\ n-1$, we have

$$\begin{aligned}
|\arg g^n(z)| &\leq \sum_{k=0}^{n-1} |\arg g^{k+1}(z) - \arg g^k(z)| + |\arg z| \\
&\leq \sum_{k=0}^{n-1} \left|\arg\left(\frac{g^{k+1}(z)}{g^k(z)}\right)\right| + \varphi \\
&= \sum_{k=0}^{n-1} |\arg(2 + \exp(-(g^k(z))^p))| + \varphi
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=0}^{n-1} c|\exp(-(g^k(z))^p)| + \varphi \\
&= \sum_{k=0}^{n-1} c\exp(-\mathrm{Re}\ (g^k(z))^p) + \varphi \\
&\leq \sum_{k=0}^{n-1} c\exp\left(-\frac{3}{2}|g^k(z)|\right) + \varphi.
\end{aligned}
$$

Since $g^k(z) \in A$ for $k = 0,\ 1, \cdots,\ n-1$, we have

$$|g^k(z)| = |g^{k-1}(z)||2 + \exp(-(g^{k-1}(z))^p)| \geq \frac{3}{2}|g^{k-1}(z)|$$

and hence we obtain

$$|\arg g^n(z)| \leq \sum_{k=0}^{n-1} c\exp\left(-\left(\frac{3}{2}\right)^{k+1}|R_2|\right) + \varphi < \theta$$

and

$$|g^n(z)| \geq \left(\frac{3}{2}\right)^n R_2.$$

Hence, we have $g^n(z) \in A$. This implies that $\{g^n\}$ converges to ∞ uniformly on compact sets in B_0.

Set $\phi_k(z) = \exp(2\pi ki/p)z$ for $k = 1,\ 2, \cdots,\ p-1$ and $B_k = \phi_k(B_0)$. Since

$$\phi_k g \phi_k^{-1}(z) = g(z),$$

we see that $\{g^n\}$ converges to ∞ uniformly on compact sets in B_k. Let F_k be the component of $F(g)$ containing B_k. Hence F_k is a Baker domain and simply connected, because F_k is unbounded. If $F_s = F_t$ for some s and t with $s \neq t$, then we see that $F_0 = F_1 = \cdots = F_{p-1}$, because of $\phi_k g \phi_k^{-1}(z) = g(z)$. Further, it is easy to see that there exists a closed curve in F_0 whose bounded complementary component contains the origin 0. On the other hand, 0 is a repelling fixed point of g and thus it is in $J(f)$. Hence F_0 is not simply connected. This contradicts that every unbounded Fatou component is simply connected for dynamics of every transcendental entire functions. Therefore $F(f)$ has at least p Baker domains F_0, $F_1, \cdots$, and F_{p-1}.

Since $f(z) = \exp(2\pi i/p)g(z)$, it is clear that

$$f^p(z) = g^p(z).$$

From the fact

$$F(f) = F(f^p) = F(g^p) = F(g)$$

we see that F_0, $F_1, \cdots$ and F_{p-1} are cyclic Baker domains of f with period p. ■

References

[1] A. F. Beardon, Iteration of Rational Functions, GTM 132, Springer, 1991.

[2] A. E. Eremenko & M. Yu. Lyubich, Dynamical properties of some classes of entire functions, Ann. Inst. Fourier Grenoble, 42(1992), 989-1020.

[3] S. Morosawa, Y. Nishimura, M. Taniguchi & T. Ueda, Holomorphic dynamics, to appear.

[4] D. Sullivan, Quasiconformal homeomorphisms and dynamics I. Solution of Fatou-Julia problem on no wandering domains. Ann of Math., 122(1985), 401-418.

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