Note on the iteration of $f_\mu(z) = z\exp(z+\mu)$

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Abstract

We show that the set of real parameters μ's for each of which the Julia set of f_μ is the whole complex plain has positive Lebesgue measure.

1 Introduction and the result

In this note, we consider a family of transcendental entire functions

$$\{f_\mu(z) = z\exp(z+\mu) \mid \mu \in \mathbf{R}\}.$$

We denote the Fatou set of f_μ and the Julia set of f_μ by $F(\mu)$ and $J(\mu)$, respectively. It is easy to see that if $\mu < 0$, then $z = 0$ is an attracting fixed point of f_μ and therefore $F(\mu) \neq \emptyset$. Conversely, as a classical problem, an existence of μ for which $F(\mu) = \emptyset$, or equivalently $J(\mu) = \mathbf{C}$, was considered. Baker ([1]) proved an existence of such μ. Further, Jang ([6]) showed that there are infinitely many such μ. Successively, Kuroda and Jang ([7]) constructed a sequence $\{\mu_n\}$ of such μ with $\mu_n \to \infty$ as $n \to \infty$. In this note, we prove the following.

Theorem 1 *Let*

$$E = \{\mu \in \mathbf{R} \mid J(\mu) = \mathbf{C}\}.$$

Then the one-dimensional Lebesgue measure of E is positive.

We now state briefly some properties of the iteration of transcendental entire functions. A cyclic component of the Fatou set is one of the following four components;
(a) an attracting component
(b) a parabolic component
(c) the Siegel disk or
(d) the Baker domain.

A singular value of a function is either a critical value of it or an asymptotic value of it. Singular values play important roles of the study of the iteration theory. The immediate basin of an attracting cycle contains at least one singular value. Same is true for a parabolic cycle. The boundary of a Siegel disk is contained in the closure of the forward orbits of the singular values. No existence of wandering component for the iteration of rational functions was proved by Sullivan ([8]). If a transcendental entire function has finitely many singular values, it is called of finite type. Eremenko and Lyubich ([3]) extended a Sullivan's result for transcendental entire functions of finite type. That is, the Fatou set of a transcendental entire function of finite type contains no wandering domain. Further, they also proved that the Fatou set of a transcendental entire function of finite type contains no Baker domain.

2 Some properties of $f_\mu(z) = z\exp(z+\mu)$

The singular values of f_μ are $f_\mu(-1) = -\exp(-1+\mu)$ and 0. Hence f_μ is of finite type and therefore $F(\mu)$ contains neither wandering domain nor the Baker domain. The fixed points of f_μ in $\mathbf{R}$ are 0 and $-\mu$. If $\mu < 0$ then 0 is an attracting fixed point. If $\mu = 0$ then 0 is a rationally indifferent fixed point. If $\mu > 0$ then 0 is a repelling fixed point. Using the graphical analysis, we see that $f_\mu^n(-1) \to 0$ as $n \to \infty$ for $\mu < 0$. Hence if $\mu < 0$ then the component of $F(\mu)$ containing 0 is the only one invariant component. We also see that $-\mu$ is an attracting fixed point for $0 < \mu < 2$ and that $-\mu$ is a rationally indifferent fixed point for $\mu = 2$. Hence we have $J(\mu) \neq \mathbf{C}$ for $\mu \leq 2$.

Hereafter, we consider the case for $\mu > 2$. Note that the orbit of $f_\mu(-1)$ is contained in the negative real axis, furthermore

$$\{f_\mu^n(-1)\}_{n=1}^{\infty} \subset [f_\mu(-1),\ 0].$$

Hence $F(\mu)$ does not contain any Siegel disk.

3 Unimodal map

We recall that f_μ is a unimodal map on $(-\infty,\ 0]$. There is a famous Jakobson's Theorem on unimodal maps.

Theorem 2 ([5]) *Let*

$$h_a(x) = ax(1-x) \qquad (0 < a \leq 4)$$

defined on $[0,\ 1]$ *and*

$$\Lambda = \left\{ a \;\middle|\; \begin{array}{l} h_a \text{ has an invariant measure on } [0,\ 1] \text{ which is absolutely} \\ \text{continuous with respect to the Lebesgue measure.} \end{array} \right\}$$

Then the Lebesgue measure of Λ *is positive.*

Consider h_a as a complex variable function. If h_a has an attracting cyclic point or a parabolic cyclic point in $\mathbf{C}$, then the intersection of its basin and the real axis is not empty, since its finite critical value is in $\mathbf{R}$. Moreover, the intersection of its basin and $[0,\ 1]$ is an open and dense set in $[0,\ 1]$. Hence, in this case, there is an invariant measure on $[0,\ 1]$ which is not absolutely continuous with respect to the Lebesgue measure. Thus, for each $a \in \Lambda$, the Fatou set of h_a coincide with the immediate basin of the super-attracting fixed point ∞.

After Jakobson, there have been many works on generalization of his result and simplification of its proof. We use one of these results proved by Tsujii([10]).

Let $k_\alpha(x) = -\alpha x$ and $\alpha = \exp(-1+\mu)$. Since $\mu > 2$, we have $\alpha > 1$. Set

$$g_\alpha(x) = k_\alpha^{-1} f_\mu k_\alpha(x) = \alpha x \exp(-\alpha x + 1).$$

Then g_α is a unimodal map from $[0,\ 1]$ to itself. Hence we consider g_α as a real valued function on $[0,\ 1]$. The set of the singular values of g_α is $\{0,\ 1\}$ and the set of the fixed points of g_α is $\{0,\ (1+\log\alpha)/\alpha\}$. Since the Schwarzian derivative of g_α is

$$Sg_\alpha(x) = -\frac{\alpha^2}{2(1-\alpha x)^2}\{(\alpha x - 2)^2 + 2\} < 0,$$

g_α is an S-unimodal map.

Furthermore, by induction, we have the following;

$$g_\alpha^n(x) = \alpha^n x \exp\left\{\alpha \sum_{k=0}^{n-1}\left(\frac{1}{\alpha} - g_\alpha^k(x)\right)\right\} \tag{1}$$

$$(g_\alpha^n)'(x) = g_\alpha^n(x)\frac{g_\alpha'(x)}{g_\alpha(x)}\alpha^{n-1}\prod_{k=1}^{n-1}\left(\frac{1}{\alpha} - g_\alpha^k(x)\right) \tag{2}$$

We now prepare a theorem on unimodal maps we need to prove Theorem 1. Let $h_t(x) = H(x,\ t)$, $x \in [0,\ 1]$, $t \in [0,\ 1]$ and each h_t is a C^2-unimodal map from $[0,\ 1]$ to itself. Set $h = h_0$ and let c be a critical point of h. We consider the following conditions;

(ND) $h''(c) \neq 0$.

(CE) There are constants a and b such that, for all $n \geq 0$,

$$\begin{aligned} &\text{(i)} \quad |dh^n(f(c))| > \exp(an - b) \\ &\text{(ii)} \quad |dh^n(x)| > \exp(an - b) \qquad \text{for } x \in h^{-n}(c) \end{aligned}$$

(Hyp) All the cyclic points of h are repelling.

(W) $\liminf_{n\to\infty} \frac{1}{n}\log|dh(h^n(c))| = 0$

(NV$_t$) $\sum_{j=0}^{\infty} \frac{\partial H(h_t^j(c),\, t)}{dh_t^j(h_t(c))} \neq 0$

Further, we define

$$\Lambda = \{t \mid h_t \text{ satisfies the conditions (ND), (CE), (Hyp), (W) and (NV}_t\text{) holds.}\}.$$

Theorem 3 ([10]) *Let Λ be the set above. Then the Lebesgue measure of Λ is positive and, for each $t \in \Lambda$, h_t has an invariant measure which is absolutely continuous with respect to the Lebesgue measure.*

In fact, Tsujii([10]) proved not only families of unimodal maps but also families of more general functions.

4 Proof of Theorem 1

Using Theorem 3, we prove Theorem 1. Let u be the maximal one satisfying that after three iteration of f_u, the critical point $1/u$ mapped to the fixed point $(1+\log u)/u$ for the first time. That is,

$$\begin{aligned} g_u^3\left(\frac{1}{u}\right) &= \frac{1}{u}(1+\log u) \quad \text{and} \\ g_u^j\left(\frac{1}{u}\right) &\neq \frac{1}{u}(1+\log u) \quad (j = 0,\ 1,\ 2) \end{aligned}$$

hold. In [7], it was proved the existence of such u and it satisfies $u > 3$. We investigate whether these conditions (ND), (CE), (Hyp), (W) and (NV$_t$) are satisfied by this u.

(ND) It is clear.

(CE) First we show (i). We have

$$g_u^n\left(\frac{1}{u}\right) = \frac{1}{u}(1+\log u)$$

for $n \geq 3$. Since $g_u(1/u) = 1$, by using (2),

$$(g_u^n)'(1) = \frac{1}{u}(1+\log u)\frac{g_u'(1)}{g_u(1)}u^{n-1}\left(\frac{1}{u} - g_u(1)\right)\prod_{k=2}^{n-1}\frac{-\log u}{u}$$

holds for $n \geq 3$. Easy calculation shows that $(g_u^0)'(1) \neq 0$, $(g_u^1)'(1) \neq 0$ and $(g_2^2)'(1) \neq 0$. It follows that (i). Facts that g_α is S-unimodal and that (i) holds imply that (ii) holds.

(Hyp) Since the asymptotic value 0 is a repelling fixed point and the critical value -1 is preperiodic, it follows that all the cyclic points are repelling.

(W) Since $g_u(1/u) = (1+\log u)/u$ for all $n \geq 3$, the equality holds.

(NV_t) From the equations (1) and (2), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\partial_\alpha g_u\left(g_u^n\left(\frac{1}{u}\right)\right)}{dg_u\left(\frac{1}{u}\right)} &= \frac{1}{u} + \frac{1}{u(1-u)} \sum_{n=0}^{\infty} (-\log u)^{-n} \\ &= \frac{1}{u} + \frac{1}{u(1-u)} \frac{\log u}{1+\log u}.\end{aligned}$$

Here, A fact $u > 3$ implies that

$$\left|\frac{1}{1-u}\right| < \frac{1}{2} \quad \text{and} \quad 0 < \frac{\log u}{1+\log u} < 1.$$

Hence, we have

$$\left|\sum_{n=0}^{\infty} \frac{\partial_\alpha g_u\left(g_u^n\left(\frac{1}{u}\right)\right)}{dg_u\left(\frac{1}{u}\right)}\right| > 0.$$

Therefore, all the conditions of Theorem 3 are satisfied for this u. Hence the set of the parameters μ's for each of which f_μ has an invariant measure with respect to Lebesgue measure has a positive measure. By the argument similar to that stated just after Theorem 2, there is neither an attracting cycle nor a parabolic cycle concerning with the critical value 1. Further, asymptotic value 0 is a repelling fixed point of g_α for $\alpha > 1$. Hence, our claim is obtained.

5 Remarks

If we pay attention to the graphs of f_μ in $(-\infty,\ 0]$, the Kuroda-Jang Theorem remind us a problem on bifurcations of real quadratic polynomials. We have a problem similar to one of real quadratic polynomials.

Problem 1 *Is the set of μ's for each of which f_μ is hyperbolic open and dense in* $\mathbf{R}$ *?*

It is easy to see that the set is open. In the case of the family of real quadratic polynomials, Świątek([9]) proved such set is dense.

Similarities between the quadratic family and our family considered in this note let us think some problem in the case that μ is complex. Set

$$f_\lambda(z) = \lambda z \exp(z).$$

Drawing the set of λ for which $\{f_\lambda^n(-1)\}_{n=1}^\infty$ is a bounded set by a computer, we can see two Mandelbrot-like sets stretched to the infinity along the real axis adjoining at the origin. In fact, Fagella studied this family in [4].

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