Non-trivial deformation of an entire function $abz + e^{bz} + c$

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Abstract

In this note, we consider dynamics of the family of transcendental entire functions $f(z; a, b, c) = abz + e^{bz} + c$. In particular, studying the case where they have a Baker domains, we investigate the area of Julia sets and the Hausdorff convergence of Julia sets.

Keywords: transcendental entire function; Baker domain; Julia set; Hausdorff convergence

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Consider the family

$$f(z; a, b, c) = abz + e^{bz} + c.$$

These $f(z; a, b, c)$ with $a, b \neq 0$ are structurally infinite ([7]), for they have infinitely many critical values. For the sake of simplicity, we assume that a, b and c are real.

Theorem 1 *Let* $ab \geq 1$, $a > 0$, $c \leq -1$ *and*

$$a \log a - a + c < 0,$$

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then $f = f(z; a, b, c)$ *has a Baker domain* $B = B_{a,b,c}$, *which is completely invariant.*

In particular, The Julia set is the boundary of B.

Proof. We set $D_0 = \{\mathrm{Re}\, z < 0\}$. Elementary calculation shows that D_0 is mapped by f to a proper subdomain in D_0, and hence D_0 is invariant. The Fatou component B containing D_0 is a Baker domain, for if $z \in D_0$, then

$$\mathrm{Re}\, f(z) < ab\mathrm{Re}\, z.$$

Also the line $L_k = \{z = x + iy \mid x \geq 0, y = (2k+1)\pi/b\}$ for $k \in \mathbb{Z}$ is mapped by f into D_0, since the assumptions imply that

$$\mathrm{Re}\, (abz + e^{bz} + c) = abx - e^{bx} + c < 0,$$

for every $z \in L_k$.

Since every preimage of D_0 by f contains either D_0 or some L_k, it intersects with D_0, which shows that B is completely invariant. ∎

The set D_0 in the proof is called a fundamental set of the Baker domain B (see [1]).

The function $f = f(z; a, b, c)$ as above has a completely invariant Baker domain B containing the left half plane and the rays L_k. Since, for the set of all the critical points,

$$\left\{ z_m = \frac{\log a + (2m+1)\pi i}{b} \right\}_{m \in \mathbb{Z}} \subset D_0 \cup \{L_k\}_{k \in \mathbb{Z}},$$

the orbits of every critical point have the imaginary parts tending to $-\infty$. Hence, a theorem of Hinkkanen, Krauskopf and Kriete [2] (essentially due to Bergweiler [3]) gives the following

Theorem 2 *For such an* $f(z; a, b, c)$ *as in Theorem 1,*

$$F(f) = B.$$

Proof. First, the boundary of the Baker domain B has a positive Euclidean distance, say $\sigma > 0$, from the post-critical set C^+, consisting of the forward orbits of all critical points of f.

On the other hand, since B contains all L_k, every other Fatou component U has a bounded distance, say η, from the boundary of B. Then a theorem of Hinkkanen, Krauskopf and Kriete gives the assertion.

But for the sake of convenience, we include a proof, following their preprint. Suppose that there were a Fatou component U, which should be either a wandering domain or a Baker domain disjoint from B. Fix a point $\zeta \in U$, and set $\zeta_n = f^n(\zeta)$. Then the disk D_n with center ζ_n and radius

$$r_n = \sigma + d(\zeta_n, B)$$

satisfies that $D_n \cap C^+ = \emptyset$. Then there are a univalent branch ϕ_n of $(f^n)^{-1}$ which send ζ_n to ζ, and a point $\xi_n \in J(f)$ such that

$$|\zeta_n - \xi_n| = d(\zeta_n, B),$$

for every n. Since functions

$$g_n(z) = \phi_n(r_n \; z + \zeta_n)$$

on $\{|z| < 1\}$ map 0 to ζ and the images are disjoint from C^+, $\{g_n\}$ is a normal family. So assume that g_n converges to g uniformly on compact sets. Then the preimages w_n of $\xi'_n = \phi_n(\xi_n)$ by g_n satisfy that

$$1 > \frac{\eta}{\sigma + \eta} > |w_n| > d$$

with some $d > 0$ by Schwarz lemma, since $\xi'_n \in J(f)$ can not tend to ζ. Thus g is non-constant and hence a univalent map, and the Carathéodory convergence theorem implies that, for an $r < 1$ sufficiently near to 1, $\{f^n\}$ on $D = g(\{|z| < r\})$ converges uniformly, and hence $D \subset F(f)$. But this contradicts the fact that $\xi'_n \in J(f)$ are contained in D for all but finite n. ∎

Remark If every point z in a component U as in the above proof has such orbits that their real parts tending to $+\infty$, then the standard argument due to Eremenko and Lyubich gives the same contradiction.

We further note the following facts.

Theorem 3 *The Julia set of such an $f(z; a, b, c)$ as in Theorem 1 has vanishing area*

Proof. We can find an $\epsilon > 0$ such that

$$(L_k)_\epsilon = \{w \in \mathbb{C} \mid d(w, L_k) \leq \epsilon\} \subset B$$

for every k, the assertion follows from McMullen-Stallard theorem in [6]. ∎

Proposition 4 *Set $a = e^{i\alpha}$ and $b = i\beta$. Then we have*

$$f(z;a,b,c) = \int_0^z 2be^{(\beta t+\alpha)i/2}\frac{\beta t-\alpha+\pi}{2}\prod_{k=1}^{\infty}\left(1-\frac{(\beta t-\alpha+\pi)^2}{4k^2\pi^2}\right)dt + d$$

Proof. Since

$$f'(z) = 2be^{(\beta z+\alpha)i/2}\cos\frac{\beta z-\alpha}{2},$$

by using the famous equation

$$\sin\pi z = \pi z\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right),$$

we have

$$f'(z) = 2be^{(\beta z+\alpha)i/2}\frac{\beta z-\alpha+\pi}{2}\prod_{k=1}^{\infty}\left(1-\frac{(\beta z-\alpha+\pi)^2}{4k^2\pi^2}\right).$$

Hence we obtain the desired equation. ■

Finally we consider convergence of Julia sets for two cases.

Theorem 5 *Fix $b > 0$ and $c < -3$ satisfying $1 - bc + \log b > 0$ and let $ab < 1$. Then the sequence of the Julia sets $f_a(z) = f(z;a,b,c)$ converges to the Julia set of $f(z) = z + e^{bz} + c$ in the sense of the Hausdorff convergence as $ab \nearrow 1$.*

Proof. The assumption $1 - bc + \log b > 0$ implies $F(f)$ consists of only one Baker domain, which can be seen by the same argument as in the proof of Theorem 1.

In the case $ab < 1$, there exists an attracting fixed point on the negative real axis, say $-r_a$ ($r_a > 0$). It is easy to see that r_a tends to infinity monotonically as $ab \nearrow 1$. We set

$$D_a = \{|z + r_a| < r_a\}.$$

For $z \in D_a$, we have

$$|f_a(z) + r_a| < abr_a + 2 < r_a + c + 3 < r_a.$$

This means D_a is a collapsing disk for the attracting fixed point $-r_a$. The sequence $\{D_a\}$ is monotonically increasing and converges to $D_0 = \{\mathrm{Re}\, z < 0\}$ in the sense of the Carathéodory convergence. To see the sequence of the basins B_a of the attracting fixed points $-r_a$ to $F(f)$ in the sense of the Carathéodory convergence, we need to show only that, for an arbitrary compact set $E \subset F(f)$, there exists an $A > 0$ such that $E \subset B_a$ for all $a > A$ with $ab < 1$.

There exists an n such that $f^n(E) \subset D_0$, for D_0 is a fundamental set. Since $\{f_a\}$ converges to f uniformly on compact sets as $ab \nearrow 1$, we have $f^n(E) \subset D_a$, and hence E is contained in B_a for all $a > A$ for some A with $ab < 1$ (see [5]). Therefore the sequence of the complement $\{B_a^c\}$ converges to the complement $F(f)^c = J(f)$ in the sense of the Hausdorff convergence. Since $J(f)$ has no interior points, we see that $\{J(f_a)\}$ converges to $J(f)$ in the sense of the Hausdorff convergence. ■

The Fatou set of $z \mapsto z + e^z$ contains infinitely many Baker domains, each of which is contained in the strip

$$S_m = \{|\text{Im } z - (2m-1)\pi| < \pi\}$$

for some $m \in \mathbb{Z} \setminus \{0\}$ (see [4]). The orbits of every point in the Baker domain have the real parts tending to $-\infty$.

Theorem 6 *Let $f_c(z) = z + e^z + c$ with $c < 0$, then $F(f_c)$ consists of only one Baker Domain. Furthermore $\{J(f_c)\}$ converges to $J(z + e^z)$ as $c \nearrow 0$ in the sense of the Hausdorff convergence.*

Proof. We set $f(z) = z + e^z$. Let B_m be the Baker domain in S_m. If $z = x + i\{(2m-1)\pi \pm (\alpha + \pi/2)\}$ with $0 \leq \alpha < \pi/2$, then we have

$$\text{Re } f(z) = x + e^x \sin\alpha > x.$$

If $z = x + i\{(2m-1)\pi \pm \alpha\}$ with $0 \leq \alpha < \pi/2$ and $x < 0$, then we have

$$\text{Im } f(z) = (2m-1)\pi \pm \alpha \mp e^x \sin\alpha$$

and thus

$$(2m-1)\pi - \alpha < \text{Im } f(z) < (2m-1)\pi + \alpha.$$

Hence, for $z \in B_m$, there exists an N such that

$$|\text{Im } f^n(z) - (2m-1)\pi| < \frac{\pi}{2}$$

for all $n > N$. Setting

$$K_m = \left\{|\text{Im } z - (2m-1)\pi| < \frac{\pi}{2},\ \text{Re } z < 0\right\},$$

we see that K_m is a fundamental set of B_m.

We consider only the case $c > -1$. Let $D_0 = \{\text{Re } z < \log(-c)\}$, then it is a fundamental set of a Baker domain of f_c. Since $f_c(z) = f(z) + c$,

$$D_0 \bigcup \left(\bigcup_{m \in \mathbb{Z} \setminus \{0\}} K_m \right)$$

is also a fundamental set. By the argument similar to that in the proof of Theorem 1, we see that $F(f_c)$ consists of only one Baker domain. Further by the argument similar to that in the proof of Theorem 5, we see that $\{J(f_c)\}$ converges to $J(f)$ as $c \nearrow 0$ in the sense of the Hausdorff convergence. ∎

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